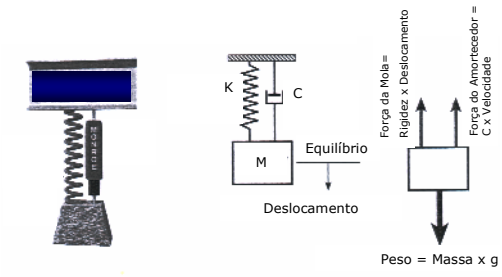


Cap. IV– Sistemas de 1 Grau de Liberdade
Vibração Livre com Amortecimento

Vibração Livre com Amortecimento



$$(\text{massa} \times \text{aceleração}) = -(\text{rigidez} \times \text{deslocamento}) - (\text{coef. amortecimento} \times \text{velocidade})$$

$$(\text{massa} \times \text{aceleração}) + (\text{coef. amortecimento} \times \text{velocidade}) + (\text{rigidez} \times \text{deslocamento}) = 0$$

Vibração Livre com Amortecimento

$$(m \times A) + (c \times V) + (k \times D) = 0$$

$$D = Xe^{st} \quad \text{onde : } s \in \text{complexos}$$

$$(m \times s^2 \times Xe^{st}) + (c \times s \times Xe^{st}) + (k \times Xe^{st}) = 0$$

$$Xe^{st} \times \left[s^2 + \left(\frac{c}{m} \right) \times s + \frac{k}{m} \right] = 0$$

$$\left[s^2 + \left(\frac{c}{m} \right) \times s + \frac{k}{m} \right] = 0$$

$$s_{1,2} = -\left(\frac{c}{2m} \right) \pm \sqrt{\left(\frac{c}{2m} \right)^2 - \frac{k}{m}} \quad \Rightarrow \quad D = A \times e^{s_1 t} + B \times e^{s_2 t}$$

Vibração Livre com Amortecimento

$$\text{Para : } \sqrt{\left(\frac{c}{2m} \right)^2 - \frac{k}{m}} = 0 \qquad s_{1,2} = s = -\left(\frac{c}{2m} \right) \quad \Rightarrow \quad D = (A + B) \times e^{\left(-\frac{c}{2m} \right) t}$$

$$\sqrt{\left(\frac{c}{2m} \right)^2 - \frac{k}{m}} = 0 \Rightarrow \left(\frac{c}{2m} \right)^2 - \frac{k}{m} = 0$$

$$\frac{c}{2m} = \sqrt{\frac{k}{m}} = \omega_n \Rightarrow c = c_{crit} = 2m\omega_n \qquad c_{crítico} = 2m\omega_n = 2\sqrt{km} \qquad \zeta = \frac{c}{c_c}$$

$$\zeta = \frac{c}{2m\omega_n} \Rightarrow \frac{c}{2m} = \omega_n \times \zeta \quad \Rightarrow \quad s_{1,2} = -\omega_n \times \zeta \pm \sqrt{(\omega_n \times \zeta)^2 - \omega_n^2}$$

Vibração Livre com Amortecimento

$$s_{1,2} = -\omega_n \times \zeta \pm \sqrt{(\omega_n \times \zeta)^2 - \omega_n^2} \qquad s_{1,2} = -\omega_n \times \zeta \pm \omega_n \sqrt{\zeta^2 - 1}$$

$$s_{1,2} = -\omega_n \times \left[\zeta \pm \sqrt{\zeta^2 - 1} \right]$$

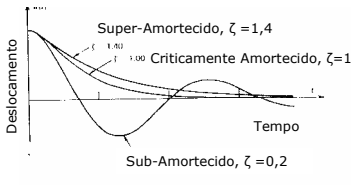
$$D = A \times e^{s_1 t} + B \times e^{s_2 t}$$

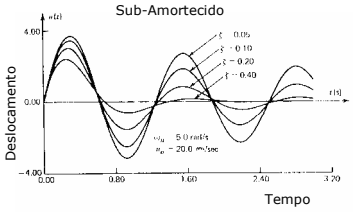
para $c = c_c$, $\zeta = 1$: $s_{1,2} = -\omega_n \Rightarrow D = (A + B) \times e^{-\omega_n t}$ Amortecimento Crítico

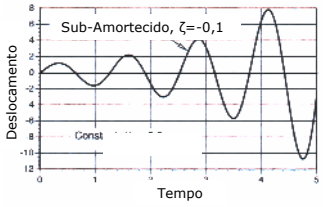
para $c > c_c$, $\zeta > 1$: $\Rightarrow D = e^{-\zeta \omega_n t} \times \left[A \times e^{-\omega_n t \times \left[\sqrt{\zeta^2 - 1}\right]} + B \times e^{\omega_n t \times \left[\sqrt{\zeta^2 - 1}\right]} \right]$ Super-Amortecido

para $c < c_c$, $\zeta < 1$: $\Rightarrow D = e^{-\zeta \omega_n t} \times \left[A \times e^{-\omega_n t \times \left[i\sqrt{1 - \zeta^2}\right]} + B \times e^{\omega_n t \times \left[i\sqrt{1 - \zeta^2}\right]} \right]$ Sub-Amortecido

Vibração Livre com Amortecimento







Vibração Livre com Amortecimento Sub-crítico

$$s_{1,2} = -\omega_n \times \left[\zeta \pm i\sqrt{1-\zeta^2}\right] \qquad s_{1,2} = -\omega_n \times \zeta \pm i\omega_n\sqrt{1-\zeta^2}$$

$$D = e^{-\zeta\omega_n t} \times \left[A \times e^{-\omega_n t \times \left[i\sqrt{1-\zeta^2}\right]} + B \times e^{\omega_n t \times \left[i\sqrt{1-\zeta^2}\right]}\right]$$

$$\omega_d = \omega_n \times \sqrt{1-\zeta^2}$$

$$D = e^{-\zeta\omega_n t} \times \left[A \times e^{-i\omega_d t} + B \times e^{i\omega_d t}\right]$$

$$e^{i\omega_d t} = \cos\omega_d t + i\,\mathrm{sen}\,\omega_d t$$
$$e^{-i\omega_d t} = \cos\omega_d t - i\,\mathrm{sen}\,\omega_d t$$

$$D = e^{-\zeta\omega_n t} \times \left[(B+A) \times \cos\omega_d t + i(B-A) \times \mathrm{sen}\,\omega_d t\right]$$

$$D = e^{-\zeta\omega_n t} \times \left[C_1 \times \cos\omega_d t + C_2 \times \mathrm{sen}\,\omega_d t\right]$$

Vibração Livre com Amortecimento Sub-crítico

$$D = e^{-\zeta\omega_n t} \times \left[C_1 \times \cos\omega_d t + C_2 \times \mathrm{sen}\,\omega_d t\right]$$

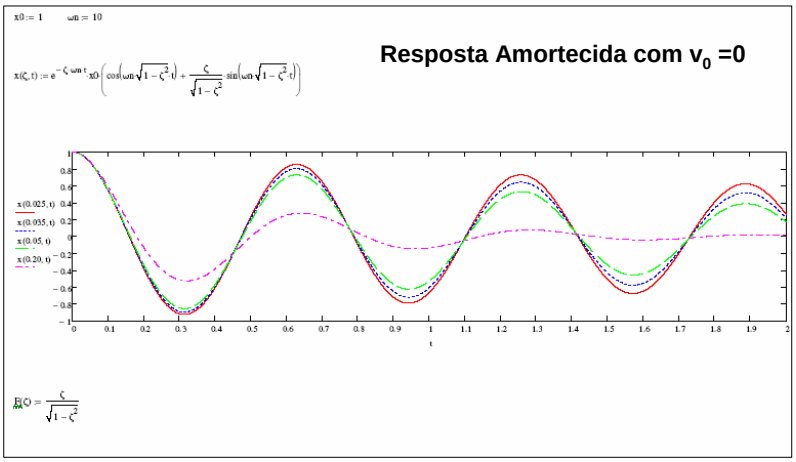
$$C_1 = D_0$$
$$C_2 = \frac{V_0 + \zeta\omega_n D_0}{\omega_d}$$

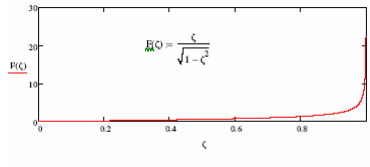
$$D = x_m e^{-\zeta\omega_n t} \times \left[\mathrm{sen}\left(\omega_d t + \phi\right)\right]$$

$$x_m = \sqrt{{D_0}^2 + \left(\frac{V_0 + \zeta\omega_n D_0}{\omega_d}\right)^2} \qquad \phi = tg^{-1}\left(\frac{\omega_d D_0}{V_0 + \zeta\omega_n D_0}\right)$$

Influência do amortecimento sobre a resposta natural (livre) do sistema:
1- Variacão na amplitude de vibração na ressonância
2- Variação no ângulo de fase na ressonância
3- Variação no valor da freqüencia de ressonância

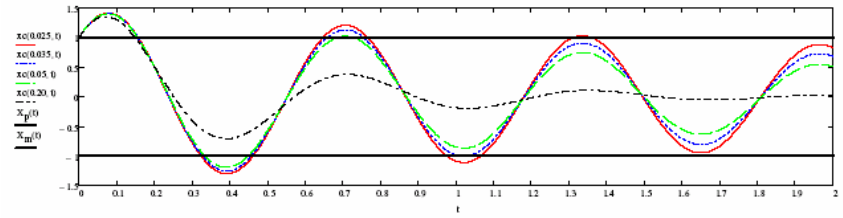
Resposta Amortecida com v_0 =0





Resposta Amortecida com v_0 >0

$$xc(\zeta, t) := e^{-\zeta\,\omega n\,t}\cdot\left(x0\cdot\cos\left(\omega n\sqrt{1-\zeta^2}\cdot t\right)+\frac{v0+\zeta\,\omega n\cdot x0}{\sqrt{1-\zeta^2}\,\omega n}\sin\left(\omega n\sqrt{1-\zeta^2}\cdot t\right)\right)$$





Vibração Livre com Amortecimento Sub-crítico

Determinação experimental de ζ:

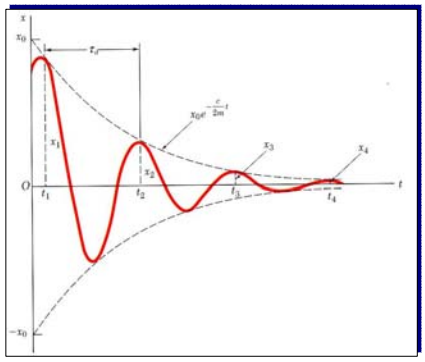
$$D = x_m e^{-\zeta\omega_n t} \times \left[\operatorname{sen}\left(\omega_d t + \phi\right)\right]$$

$$\frac{D_1}{D_2} = \frac{x_m e^{-\zeta\omega_n t_1} \times \left[\operatorname{sen}\left(\omega_d t_1 + \phi\right)\right]}{x_m e^{-\zeta\omega_n t_2} \times \left[\operatorname{sen}\left(\omega_d t_2 + \phi\right)\right]}$$

$$t_2 = t_1 + \tau_d$$

$$\text{Dec Log} = \ln\left(\frac{D_1}{D_2}\right) = 2\pi\zeta\frac{\omega_n}{\omega_d}$$

$$\omega_d = \omega_n \times \sqrt{1-\zeta^2}$$

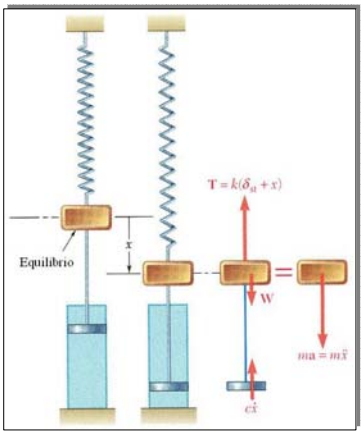


$$\ln\left(\frac{D_1}{D_2}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \quad \text{Para } c << c_c \text{ ou } \zeta << 1 \Rightarrow \quad \zeta \simeq \frac{1}{2\pi}\ln\left(\frac{D_1}{D_2}\right)$$

Para o k-ésimo ciclo:

$$\text{Dec Log} = \frac{1}{k} \times \ln\left(\frac{D_1}{D_{k+1}}\right)$$

Vibração Livre com Amortecimento



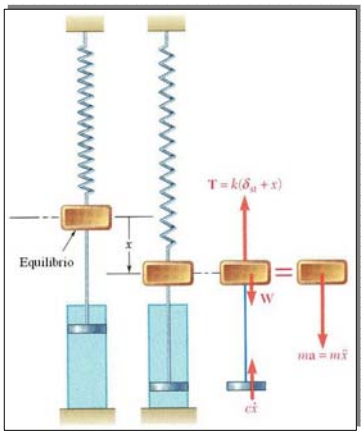
- Todas as vibrações são amortecidas em algum grau pelas forças devido a *atrito seco*, *atrito fluido*, ou *atrito interno*.
- Com *amortecimento viscoso* devido a atrito fluido,

$$+\downarrow \boxed{\sum F = ma:}$$

$$\boxed{W - k\left(\delta_{st} + x\right) - c\dot{x} = m\ddot{x}}$$

$$\boxed{m\ddot{x} + c\dot{x} + kx = 0}$$

Vibração Livre com Amortecimento



- Substituindo $x = e^{\lambda t}$ leva à *equação característica*,

$$m\lambda^2 + c\lambda + k = 0 \qquad \lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

- O coeficiente de amortecimento crítico é tal que:

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0 \qquad c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n$$

Ex. 1: O bloco mostrado na figura é abaixado em 24mm da sua posição de equilíbrio e então liberado. Sabendo que, após de 10 ciclos, o deslocamento máximo do bloco é de 10mm, determine (a) o fator de amortecimento, (b) o valor do coeficiente de amortecimento viscoso.

