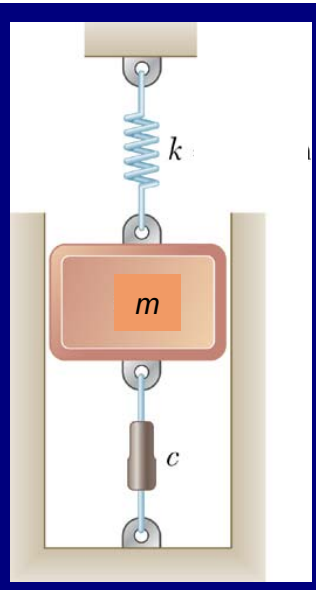
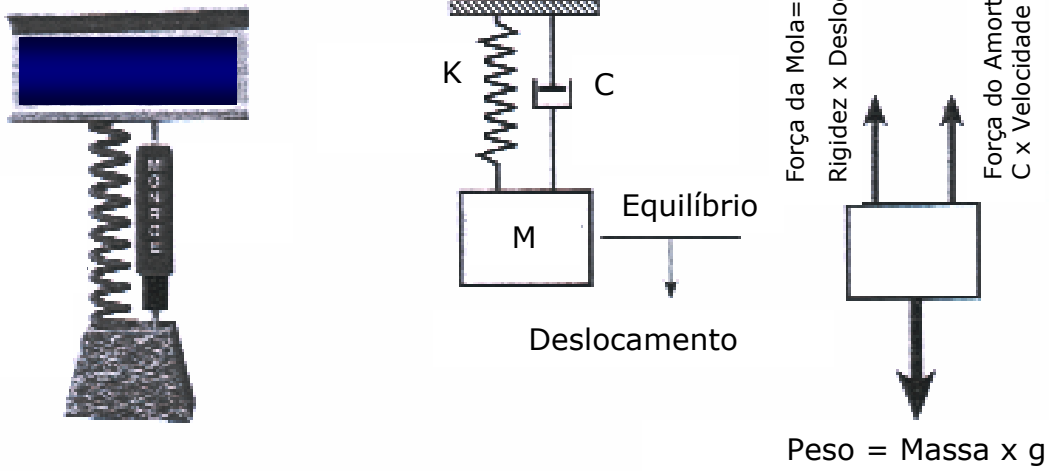


Cap. IV– Sistemas de 1 Grau de Liberdade
Vibração Livre com Amortecimento

Vibração Livre com Amortecimento



$$(\text{massa} \times \text{aceleração}) = -(\text{rigidez} \times \text{deslocamento}) - (\text{coef. amortecimento} \times \text{velocidade})$$

$$(\text{massa} \times \text{aceleração}) + (\text{coef. amortecimento} \times \text{velocidade}) + (\text{rigidez} \times \text{deslocamento}) = 0$$

Vibração Livre com Amortecimento

$$(m \times A) + (c \times V) + (k \times D) = 0$$

$$D = Xe^{st} \qquad \text{onde : } s \in \text{complexos}$$

$$(m \times s^2 \times Xe^{st}) + (c \times s \times Xe^{st}) + (k \times Xe^{st}) = 0$$

$$Xe^{st} \times \left[s^2 + \left(\frac{c}{m} \right) \times s + \frac{k}{m} \right] = 0$$

$$\left[s^2 + \left(\frac{c}{m} \right) \times s + \frac{k}{m} \right] = 0$$

$$s_{1,2} = -\left(\frac{c}{2m} \right) \pm \sqrt{\left(\frac{c}{2m} \right)^2 - \frac{k}{m}} \quad \Rightarrow \quad D = A \times e^{s_1 t} + B \times e^{s_2 t}$$

Vibração Livre com Amortecimento

$$\text{Para}: \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = 0 \qquad s_{1,2} = s = -\left(\frac{c}{2m}\right) \Rightarrow D = (A+B) \times e^{\left(-\frac{c}{2m}\right)t}$$

$$\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = 0 \Rightarrow \left(\frac{c}{2m}\right)^2 - \frac{k}{m} = 0$$

$$\frac{c}{2m} = \sqrt{\frac{k}{m}} = \omega_n \Rightarrow c = c_{crit} = 2m\omega_n \qquad c_{crítico} = 2m\omega_n = 2\sqrt{km} \qquad \zeta = \frac{c}{c_c}$$

$$\zeta = \frac{c}{2m\omega_n} \Rightarrow \frac{c}{2m} = \omega_n \times \zeta \qquad \Rightarrow \qquad s_{1,2} = -\omega_n \times \zeta \pm \sqrt{(\omega_n \times \zeta)^2 - \omega_n^2}$$

Vibração Livre com Amortecimento

$$s_{1,2} = -\omega_n \times \zeta \pm \sqrt{\left(\omega_n \times \zeta\right)^2 - \omega_n^2} \qquad s_{1,2} = -\omega_n \times \zeta \pm \omega_n \sqrt{\zeta^2 - 1}$$

$$s_{1,2} = -\omega_n \times \left[\zeta \pm \sqrt{\zeta^2 - 1}\right]$$

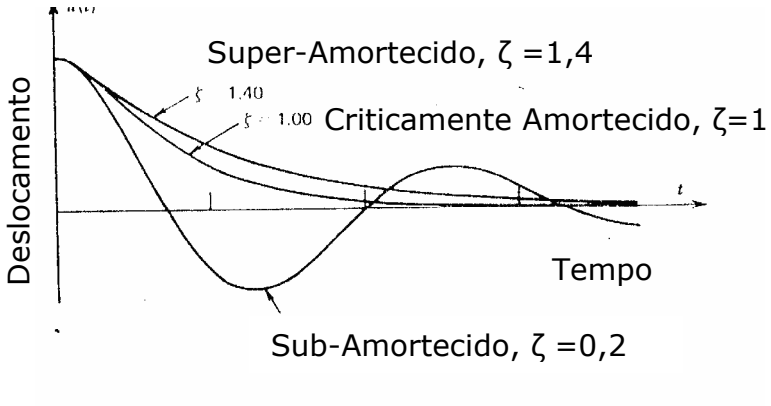
$$D = A \times e^{s_1 t} + B \times e^{s_2 t}$$

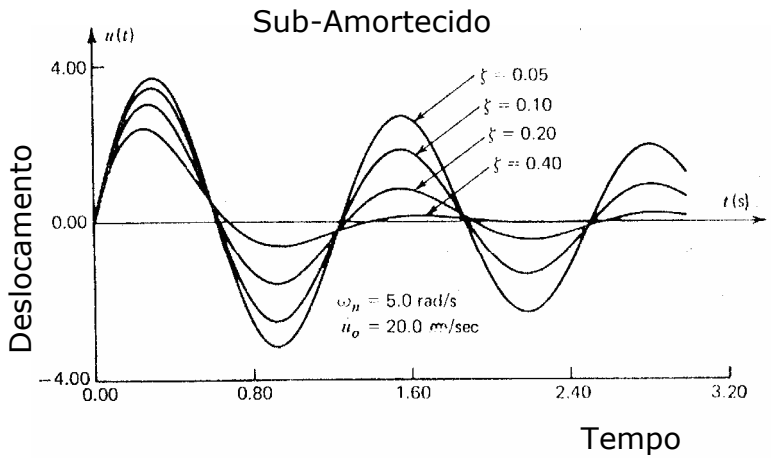
para $c = c_c$, $\zeta = 1$: $\quad s_{1,2} = -\omega_n \quad \Longrightarrow \quad D = (A + B) \times e^{-\omega_n t}$ Amortecimento Crítico

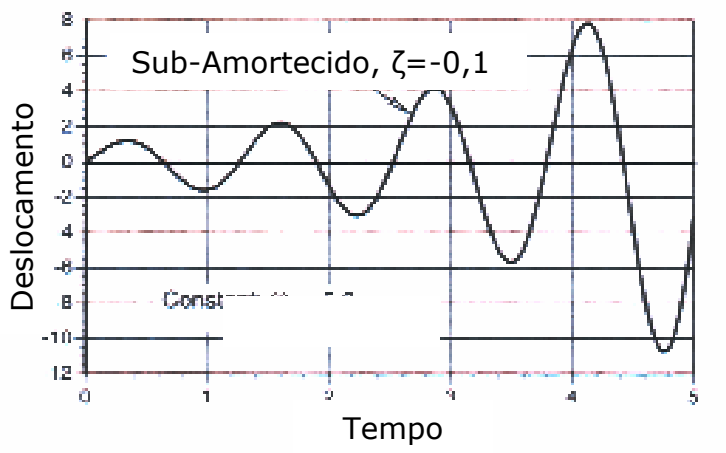
para $c > c_c$, $\zeta > 1$: $\quad \Longrightarrow D = e^{-\zeta\omega_n t} \times \left[A \times e^{-\omega_n t \times \left[\sqrt{\zeta^2 - 1}\right]} + B \times e^{\omega_n t \times \left[\sqrt{\zeta^2 - 1}\right]}\right]$ Super-Amortecido

para $c < c_c$, $\zeta < 1$: $\quad \Longrightarrow D = e^{-\zeta\omega_n t} \times \left[A \times e^{-\omega_n t \times \left[i\sqrt{1 - \zeta^2}\right]} + B \times e^{\omega_n t \times \left[i\sqrt{1 - \zeta^2}\right]}\right]$ Sub-Amortecido

Vibração Livre com Amortecimento







Vibração Livre com Amortecimento Sub-crítico

$$s_{1,2} = -\omega_n \times \left[\zeta \pm i\sqrt{1-\zeta^2} \right]$$

$$s_{1,2} = -\omega_n \times \zeta \pm i\omega_n \sqrt{1-\zeta^2}$$

$$D = e^{-\zeta\omega_n t} \times \left[A \times e^{-\omega_n t \times \left[i\sqrt{1-\zeta^2} \right]} + B \times e^{\omega_n t \times \left[i\sqrt{1-\zeta^2} \right]} \right]$$

$$\omega_d = \omega_n \times \sqrt{1-\zeta^2}$$

$$D = e^{-\zeta\omega_n t} \times \left[A \times e^{-i\omega_d t} + B \times e^{i\omega_d t} \right]$$

$$e^{i\omega_d t} = \cos\omega_d t + i\,\mathrm{sen}\,\omega_d t$$

$$e^{-i\omega_d t} = \cos\omega_d t - i\,\mathrm{sen}\,\omega_d t$$

$$D = e^{-\zeta\omega_n t} \times \left[(B+A) \times \cos\omega_d t + i(B-A) \times \mathrm{sen}\,\omega_d t \right]$$

$$D = e^{-\zeta\omega_n t} \times \left[C_1 \times \cos\omega_d t + C_2 \times \mathrm{sen}\,\omega_d t \right]$$

Vibração Livre com Amortecimento Sub-crítico

$$D = e^{-\zeta\omega_n t} \times \left[C_1 \times \cos \omega_d t + C_2 \times \text{sen}\omega_d t \right]$$

$$C_1 = D_0$$

$$C_2 = \frac{V_0 + \zeta\omega_n D_0}{\omega_d}$$

$$D = x_m e^{-\zeta\omega_n t} \times \left[\text{sen}\left(\omega_d t + \phi \right) \right]$$

$$x_m = \sqrt{D_0^{\,2} + \left(\frac{V_0 + \zeta\omega_n D_0}{\omega_d} \right)^2}$$

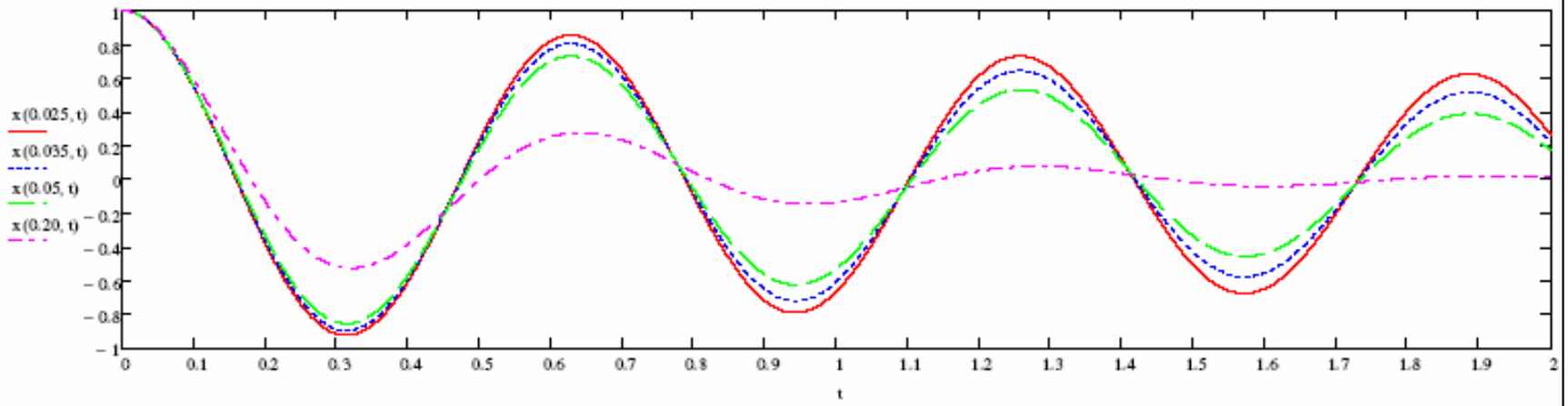
$$\phi = \text{tg}^{-1} \left(\frac{\omega_d D_0}{V_0 + \zeta\omega_n D_0} \right)$$

Influência do amortecimento sobre a resposta natural (livre) do sistema:
1- Variacão na amplitude de vibração na ressonância
2- Variação no ângulo de fase na ressonância
3- Variação no valor da freqüencia de ressonância

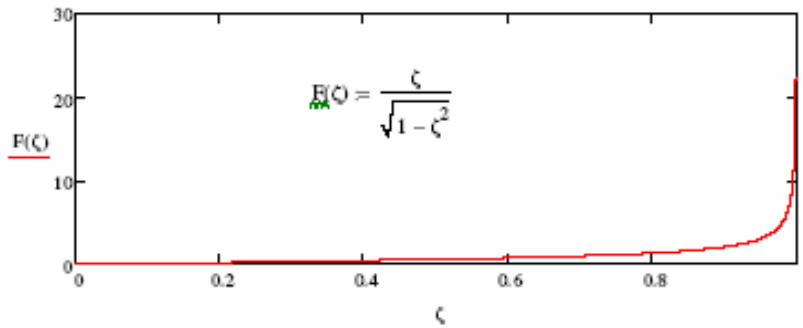
Resposta Amortecida com $v_0 = 0$

$x0 := 1 \qquad \omega n := 10$

$$x(\zeta, t) := e^{-\zeta \cdot \omega n \cdot t} \cdot x0 \cdot \left(\cos\left(\omega n \cdot \sqrt{1-\zeta^2} \cdot t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}} \cdot \sin\left(\omega n \cdot \sqrt{1-\zeta^2} \cdot t\right) \right)$$

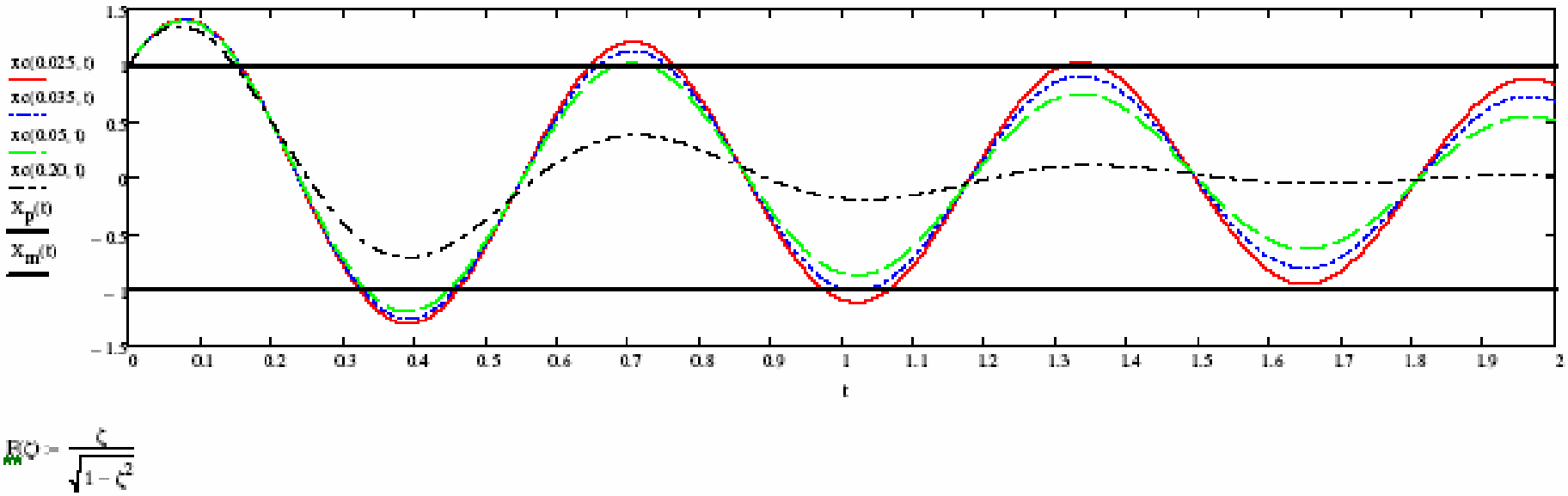


$$F(\zeta) := \frac{\zeta}{\sqrt{1-\zeta^2}}$$



Resposta Amortecida com $v_0 > 0$

$$xo(\zeta, t) := e^{-\zeta \cdot \omega n \cdot t} \cdot \left(x0 \cdot \cos\left(\omega n \cdot \sqrt{1-\zeta^2} \cdot t\right) + \frac{v0 + \zeta \cdot \omega n \cdot x0}{\sqrt{1-\zeta^2} \cdot \omega n} \cdot \sin\left(\omega n \cdot \sqrt{1-\zeta^2} \cdot t\right) \right)$$



$$EQ := \frac{\zeta}{\sqrt{1-\zeta^2}}$$

Vibração Livre com Amortecimento Sub-crítico

Determinação experimental de ζ:

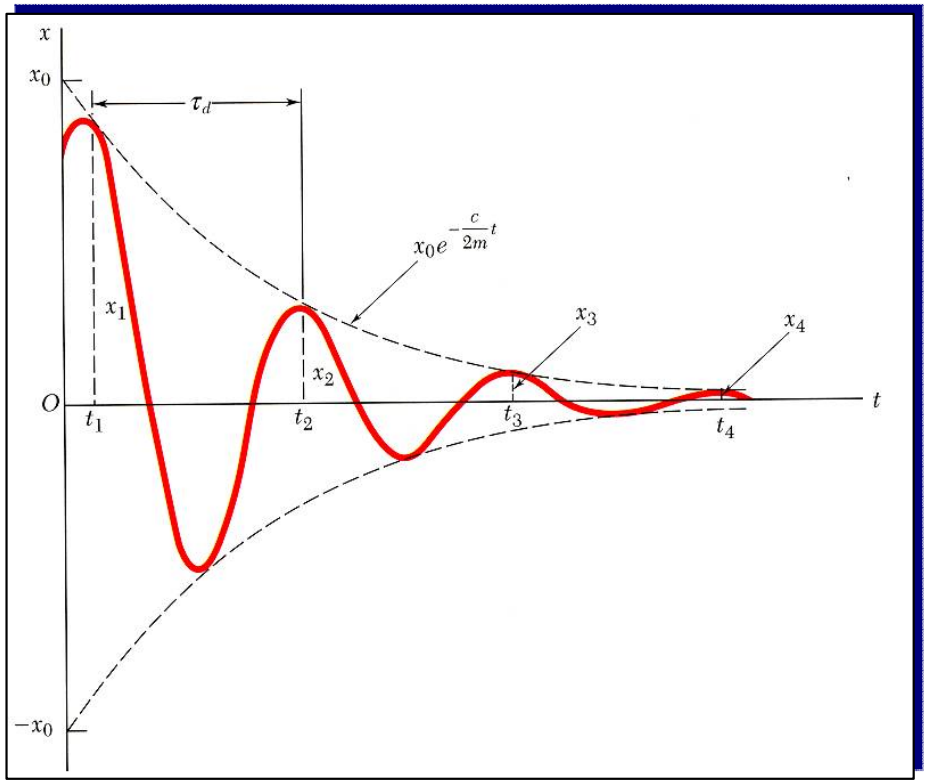
$$D = x_m e^{-\zeta\omega_n t} \times \left[\operatorname{sen}\left(\omega_d t + \phi\right)\right]$$

$$\frac{D_1}{D_2} = \frac{x_m e^{-\zeta\omega_n t_1} \times \left[\operatorname{sen}\left(\omega_d t_1 + \phi\right)\right]}{x_m e^{-\zeta\omega_n t_2} \times \left[\operatorname{sen}\left(\omega_d t_2 + \phi\right)\right]}$$

$$t_2 = t_1 + \tau_d$$

$$\text{Dec Log} = \ln\left(\frac{D_1}{D_2}\right) = 2\pi\zeta\frac{\omega_n}{\omega_d}$$

$$\omega_d = \omega_n \times \sqrt{1-\zeta^2}$$



$$\ln\left(\frac{D_1}{D_2}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

Para $c \ll c_c$

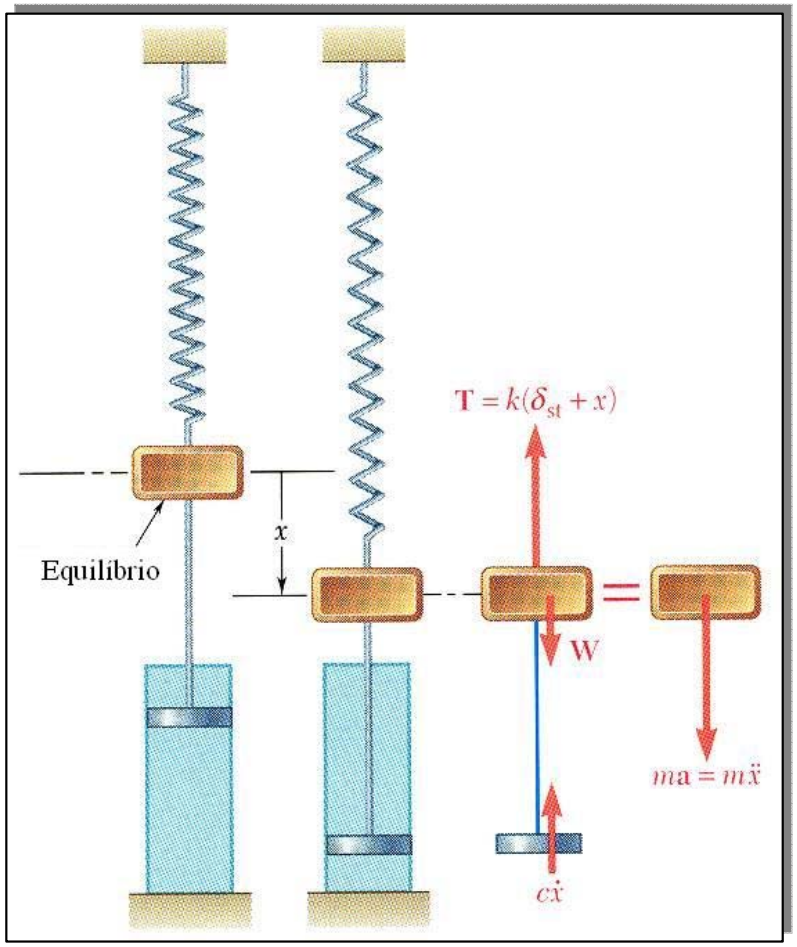
ou $\zeta \ll 1$

$$\zeta \simeq \frac{1}{2\pi}\ln\left(\frac{D_1}{D_2}\right)$$

Para o k-ésimo ciclo:

$$\text{Dec Log} = \frac{1}{k} \times \ln\left(\frac{D_1}{D_{k+1}}\right)$$

Vibração Livre com Amortecimento



- Todas as vibrações são amortecidas em algum grau pelas forças devido a *atrito seco*, *atrito fluido*, ou *atrito interno*.

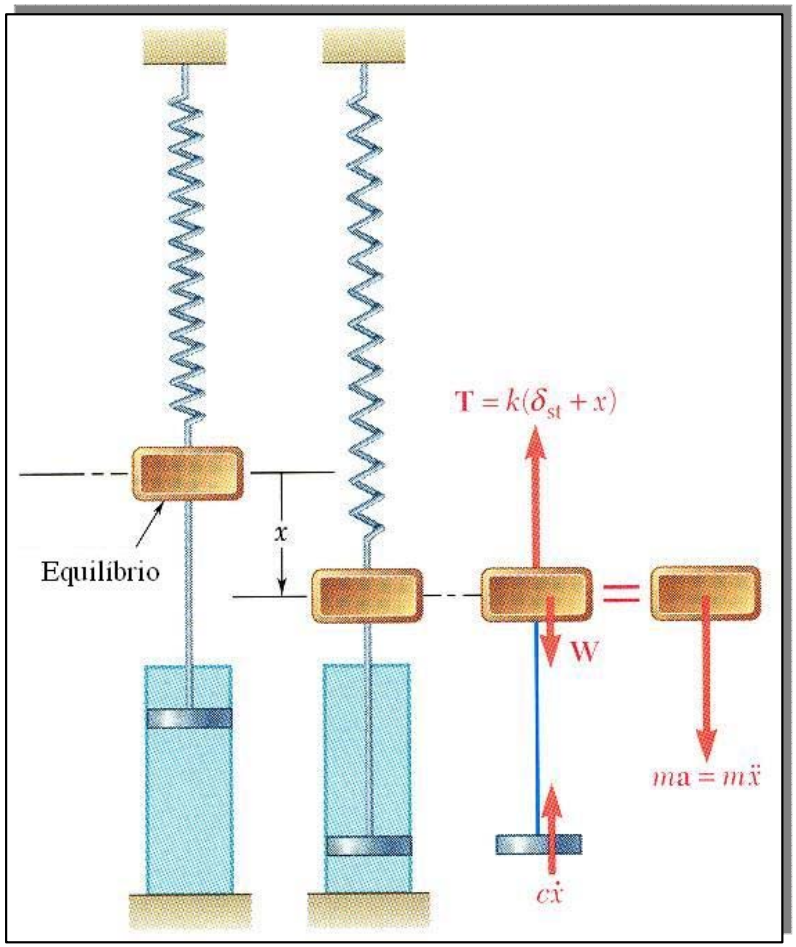
- Com *amortecimento viscoso* devido a atrito fluido,

$$+\downarrow \sum F = ma:$$

$$W - k\left(\delta_{st} + x\right) - c\dot{x} = m\ddot{x}$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

Vibração Livre com Amortecimento



- Substituindo $x = e^{\lambda t}$ leva à *equação característica*,

$$m\lambda^2 + c\lambda + k = 0 \qquad \lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

- O coeficiente de amortecimento crítico é tal que:

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0 \qquad c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n$$

Ex. 1: O bloco mostrado na figura é abaixado em 24mm da sua posição de equilíbrio e então liberado. Sabendo que, após de 10 ciclos, o deslocamento máximo do bloco é de 10mm, determine (a) o fator de amortecimento, (b) o valor do coeficiente de amortecimento viscoso.

