Part IV Fourier Series and Partial Differential Equations

12 Orthogonal Functions and Fourier Series

EXERCISES 12.1

Orthogonal Functions

1. $\displaystyle\int_{-2}^{2} xx^2 dx = \frac{1}{4}x^4 \Big|_{-2}^{2} = 0$

2. $\displaystyle\int_{-1}^{1} x^3(x^2+1)dx = \frac{1}{6}x^6 \Big|_{-1}^{1} + \frac{1}{4}x^4 \Big|_{-1}^{1} = 0$

3. $\displaystyle\int_{0}^{2} e^x(xe^{-x} - e^{-x})dx = \int_{0}^{2}(x-1)dx = \left(\frac{1}{2}x^2 - x\right)\Big|_{0}^{2} = 0$

4. $\displaystyle\int_{0}^{\pi} \cos x \sin^2 x\, dx = \frac{1}{3}\sin^3 x \Big|_{0}^{\pi} = 0$

5. $\displaystyle\int_{-\pi/2}^{\pi/2} x\cos 2x\, dx = \frac{1}{2}\left(\frac{1}{2}\cos 2x + x\sin 2x\right)\Big|_{-\pi/2}^{\pi/2} = 0$

6. $\displaystyle\int_{\pi/4}^{5\pi/4} e^x \sin x\, dx = \left(\frac{1}{2}e^x \sin x - \frac{1}{2}e^x \cos x\right)\Big|_{\pi/4}^{5\pi/4} = 0$

7. For $m \neq n$

$$\int_0^{\pi/2} \sin(2n+1)x\, \sin(2m+1)x\, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \Big(\cos 2(n-m)x - \cos 2(n+m+1)x\Big)\, dx$$

$$= \frac{1}{4(n-m)}\sin 2(n-m)x \Big|_0^{\pi/2} - \frac{1}{4(n+m+1)}\sin 2(n+m+1)x \Big|_0^{\pi/2} = 0.$$

For $m = n$

$$\int_0^{\pi/2} \sin^2(2n+1)x\,dx = \int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{2}\cos 2(2n+1)x\right) dx$$

$$= \frac{1}{2}x\,\Big|_0^{\pi/2} - \frac{1}{4(2n+1)}\sin 2(2n+1)x\,\Big|_0^{\pi/2} = \frac{\pi}{4}$$

so that

$$\|\sin(2n+1)x\| = \frac{1}{2}\sqrt{\pi}\,.$$

8. For $m \neq n$

$$\int_0^{\pi/2} \cos(2n+1)x\,\cos(2m+1)x\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \Big(\cos 2(n-m)x + \cos 2(n+m+1)x\Big)\,dx$$

$$= \frac{1}{4(n-m)}\sin 2(n-m)x\,\Big|_0^{\pi/2} + \frac{1}{4(n+m+1)}\sin 2(n+m+1)x\,\Big|_0^{\pi/2} = 0.$$

For $m = n$

$$\int_0^{\pi/2} \cos^2(2n+1)x\,dx = \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2(2n+1)x\right) dx$$

$$= \frac{1}{2}x\,\Big|_0^{\pi/2} + \frac{1}{4(2n+1)}\sin 2(2n+1)x\,\Big|_0^{\pi/2} = \frac{\pi}{4}$$

so that

$$\|\cos(2n+1)x\| = \frac{1}{2}\sqrt{\pi}\,.$$

9. For $m \neq n$

$$\int_0^{\pi} \sin nx \sin mx\,dx = \frac{1}{2}\int_0^{\pi} \Big(\cos(n-m)x - \cos(n+m)x\Big)\,dx$$

$$= \frac{1}{2(n-m)}\sin(n-m)x\,\Big|_0^{\pi} - \frac{1}{2(n+m)}\sin(n+m)x\,\Big|_0^{\pi} = 0.$$

For $m = n$

$$\int_0^{\pi} \sin^2 nx\,dx = \int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2nx\right) dx = \frac{1}{2}x\,\Big|_0^{\pi} - \frac{1}{4n}\sin 2nx\,\Big|_0^{\pi} = \frac{\pi}{2}$$

so that

$$\|\sin nx\| = \sqrt{\frac{\pi}{2}}\,.$$

10. For $m \neq n$

$$\int_0^{p} \sin\frac{n\pi}{p}x\,\sin\frac{m\pi}{p}x\,dx = \frac{1}{2}\int_0^{p} \left(\cos\frac{(n-m)\pi}{p}x - \cos\frac{(n+m)\pi}{p}x\right)\,dx$$

$$= \frac{p}{2(n-m)\pi}\sin\frac{(n-m)\pi}{p}x\,\Big|_0^{p} - \frac{p}{2(n+m)\pi}\sin\frac{(n+m)\pi}{p}x\,\Big|_0^{p} = 0.$$

For $m = n$

$$\int_0^{p} \sin^2\frac{n\pi}{p}x\,dx = \int_0^{p} \left(\frac{1}{2} - \frac{1}{2}\cos\frac{2n\pi}{p}x\right) dx = \frac{1}{2}x\,\Big|_0^{p} - \frac{p}{4n\pi}\sin\frac{2n\pi}{p}x\,\Big|_0^{p} = \frac{p}{2}$$

so that

$$\left\|\sin\frac{n\pi}{p}x\right\| = \sqrt{\frac{p}{2}}\,.$$

11. For $m \neq n$

$$\int_0^p \cos\frac{n\pi}{p}x\,\cos\frac{m\pi}{p}x\,dx = \frac{1}{2}\int_0^p \left(\cos\frac{(n-m)\pi}{p}x + \cos\frac{(n+m)\pi}{p}x\right)dx$$

$$= \frac{p}{2(n-m)\pi}\sin\frac{(n-m)\pi}{p}x\,\Big|_0^p + \frac{p}{2(n+m)\pi}\sin\frac{(n+m)\pi}{p}x\,\Big|_0^p = 0.$$

For $m = n$

$$\int_0^p \cos^2\frac{n\pi}{p}x\,dx = \int_0^p \left(\frac{1}{2} + \frac{1}{2}\cos\frac{2n\pi}{p}x\right)dx = \frac{1}{2}x\,\Big|_0^p + \frac{p}{4n\pi}\sin\frac{2n\pi}{p}x\,\Big|_0^p = \frac{p}{2}\,.$$

Also

$$\int_0^p 1\cdot\cos\frac{n\pi}{p}x\,dx = \frac{p}{n\pi}\sin\frac{n\pi}{p}x\,\Big|_0^p = 0 \quad\text{and}\quad \int_0^p 1^2 dx = p$$

so that

$$\|1\| = \sqrt{p} \quad\text{and}\quad \left\|\cos\frac{n\pi}{p}x\right\| = \sqrt{\frac{p}{2}}\,.$$

12. For $m \neq n$, we use Problems 11 and 10:

$$\int_{-p}^p \cos\frac{n\pi}{p}x\,\cos\frac{m\pi}{p}x\,dx = 2\int_0^p \cos\frac{n\pi}{p}x\,\cos\frac{m\pi}{p}x\,dx = 0$$

$$\int_{-p}^p \sin\frac{n\pi}{p}x\,\sin\frac{m\pi}{p}x\,dx = 2\int_0^p \sin\frac{n\pi}{p}x\,\sin\frac{m\pi}{p}x\,dx = 0.$$

Also

$$\int_{-p}^p \sin\frac{n\pi}{p}x\,\cos\frac{m\pi}{p}x\,dx = \frac{1}{2}\int_{-p}^p \left(\sin\frac{(n-m)\pi}{p}x + \sin\frac{(n+m)\pi}{p}x\right)dx = 0,$$

$$\int_{-p}^p 1\cdot\cos\frac{n\pi}{p}x\,dx = \frac{p}{n\pi}\sin\frac{n\pi}{p}x\,\Big|_{-p}^p = 0,$$

$$\int_{-p}^p 1\cdot\sin\frac{n\pi}{p}x\,dx = -\frac{p}{n\pi}\cos\frac{n\pi}{p}x\,\Big|_{-p}^p = 0,$$

and

$$\int_{-p}^p \sin\frac{n\pi}{p}x\,\cos\frac{n\pi}{p}x\,dx = \int_{-p}^p \frac{1}{2}\sin\frac{2n\pi}{p}x\,dx = -\frac{p}{4n\pi}\cos\frac{2n\pi}{p}x\,\Big|_{-p}^p = 0.$$

For $m = n$

$$\int_{-p}^p \cos^2\frac{n\pi}{p}x\,dx = \int_{-p}^p \left(\frac{1}{2} + \frac{1}{2}\cos\frac{2n\pi}{p}x\right)dx = p,$$

$$\int_{-p}^p \sin^2\frac{n\pi}{p}x\,dx = \int_{-p}^p \left(\frac{1}{2} - \frac{1}{2}\cos\frac{2n\pi}{p}x\right)dx = p,$$

and

$$\int_{-p}^p 1^2 dx = 2p$$

so that

$$\|1\| = \sqrt{2p}\,, \quad \left\|\cos\frac{n\pi}{p}x\right\| = \sqrt{p}\,, \quad\text{and}\quad \left\|\sin\frac{n\pi}{p}x\right\| = \sqrt{p}\,.$$

13. Since

$$\int_{-\infty}^{\infty} e^{-x^2} \cdot 1 \cdot 2x\,dx = -e^{-x^2}\Big|_{-\infty}^{0} - e^{-x^2}\Big|_{0}^{\infty} = 0,$$

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-x^2} \cdot 1 \cdot (4x^2-2)\,dx &= 2\int_{-\infty}^{\infty} x\left(2xe^{-x^2}\right)dx - 2\int_{-\infty}^{\infty} e^{-x^2}dx \\ &= 2\left(-xe^{-x^2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2}dx\right) - 2\int_{-\infty}^{\infty} e^{-x^2}dx \\ &= 2\left(-xe^{-x^2}\Big|_{-\infty}^{0} - xe^{-x^2}\Big|_{0}^{\infty}\right) = 0,\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-x^2} \cdot 2x \cdot (4x^2-2)\,dx &= 4\int_{-\infty}^{\infty} x^2\left(2xe^{-x^2}\right)dx - 4\int_{-\infty}^{\infty} xe^{-x^2}dx \\ &= 4\left(-x^2e^{-x^2}\Big|_{-\infty}^{\infty} + 2\int_{-\infty}^{\infty} xe^{-x^2}dx\right) - 4\int_{-\infty}^{\infty} xe^{-x^2}dx \\ &= 4\left(-x^2e^{-x^2}\Big|_{-\infty}^{0} - x^2e^{-x^2}\Big|_{0}^{\infty}\right) + 2\int_{-\infty}^{\infty} 2xe^{-x^2}dx = 0,\end{aligned}$$

the functions are orthogonal.

14. Since

$$\int_{0}^{\infty} e^{-x} \cdot 1(1-x)\,dx = (x-1)e^{-x}\Big|_{0}^{\infty} - \int_{0}^{\infty} e^{-x}dx = 0,$$

$$\begin{aligned}\int_{0}^{\infty} e^{-x} \cdot 1 \cdot \left(\frac{1}{2}x^2 - 2x + 1\right)dx &= \left(2x - 1 - \frac{1}{2}x^2\right)e^{-x}\Big|_{0}^{\infty} + \int_{0}^{\infty} e^{-x}(x-2)\,dx \\ &= 1 + (2-x)e^{-x}\Big|_{0}^{\infty} + \int_{0}^{\infty} e^{-x}dx = 0,\end{aligned}$$

and

$$\begin{aligned}&\int_{0}^{\infty} e^{-x} \cdot (1-x)\left(\frac{1}{2}x^2 - 2x + 1\right)dx \\ &\quad= \int_{0}^{\infty} e^{-x}\left(-\frac{1}{2}x^3 + \frac{5}{2}x^2 - 3x + 1\right)dx \\ &\quad= e^{-x}\left(\frac{1}{2}x^3 - \frac{5}{2}x^2 + 3x - 1\right)\Big|_{0}^{\infty} + \int_{0}^{\infty} e^{-x}\left(-\frac{3}{2}x^2 + 5x - 3\right)dx \\ &\quad= 1 + e^{-x}\left(\frac{3}{2}x^2 - 5x + 3\right)\Big|_{0}^{\infty} + \int_{0}^{\infty} e^{-x}(5-3x)\,dx \\ &\quad= 1 - 3 + e^{-x}(3x-5)\Big|_{0}^{\infty} - 3\int_{0}^{\infty} e^{-x}dx = 0,\end{aligned}$$

the functions are orthogonal.

15. By orthogonality $\int_a^b \phi_0(x)\phi_n(x)dx = 0$ for $n = 1, 2, 3, \ldots$; that is, $\int_a^b \phi_n(x)dx = 0$ for $n = 1, 2, 3, \ldots$.

16. Using the facts that ϕ_0 and ϕ_1 are orthogonal to ϕ_n for $n > 1$, we have

$$\begin{aligned}\int_a^b (\alpha x + \beta)\phi_n(x)\,dx &= \alpha \int_a^b x\phi_n(x)\,dx + \beta \int_a^b 1 \cdot \phi_n(x)\,dx \\ &= \alpha \int_a^b \phi_1(x)\phi_n(x)\,dx + \beta \int_a^b \phi_0(x)\phi_n(x)\,dx \\ &= \alpha \cdot 0 + \beta \cdot 0 = 0\end{aligned}$$

for $n = 2, 3, 4, \ldots$.

17. Using the fact that ϕ_n and ϕ_m are orthogonal for $n \neq m$ we have

$$\begin{aligned}\|\phi_m(x) + \phi_n(x)\|^2 &= \int_a^b [\phi_m(x) + \phi_n(x)]^2 dx = \int_a^b \left[\phi_m^2(x) + 2\phi_m(x)\phi_n(x) + \phi_n^2(x)\right] dx \\ &= \int_a^b \phi_m^2(x)dx + 2\int_a^b \phi_m(x)\phi_n(x)dx + \int_a^b \phi_n^2(x)\,dx \\ &= \|\phi_m(x)\|^2 + \|\phi_n(x)\|^2 .\end{aligned}$$

18. Setting

$$0 = \int_{-2}^{2} f_3(x)f_1(x)\,dx = \int_{-2}^{2} \left(x^2 + c_1x^3 + c_2x^4\right) dx = \frac{16}{3} + \frac{64}{5}c_2$$

and

$$0 = \int_{-2}^{2} f_3(x)f_2(x)\,dx = \int_{-2}^{2} \left(x^3 + c_1x^4 + c_2x^5\right) dx = \frac{64}{5}c_1$$

we obtain $c_1 = 0$ and $c_2 = -5/12$.

19. Since $\sin nx$ is an odd function on $[-\pi, \pi]$,

$$(1, \sin nx) = \int_{-\pi}^{\pi} \sin nx\,dx = 0$$

and $f(x) = 1$ is orthogonal to every member of $\{\sin nx\}$. Thus $\{\sin nx\}$ is not complete.

20. $(f_1 + f_2, f_3) = \displaystyle\int_a^b [f_1(x) + f_2(x)]f_3(x)\,dx = \int_a^b f_1(x)f_3(x)\,dx + \int_a^b f_2(x)f_3(x)\,dx = (f_1, f_3) + (f_2, f_3)$

21. **(a)** The fundamental period is $2\pi/2\pi = 1$.

(b) The fundamental period is $2\pi/(4/L) = \frac{1}{2}\pi L$.

(c) The fundamental period of $\sin x + \sin 2x$ is 2π.

(d) The fundamental period of $\sin 2x + \cos 4x$ is $2\pi/2 = \pi$.

(e) The fundamental period of $\sin 3x + \cos 4x$ is 2π since the smallest integer multiples of $2\pi/3$ and $2\pi/4 = \pi/2$ that are equal are 3 and 4, respectively.

(f) The fundamental period of $f(x)$ is $2\pi/(n\pi/p) = 2p/n$.

22. **(a)** Following the pattern established by $\phi_1(x)$ and $\phi_2(x)$ we have

$$\phi_3(x) = f_3(x) - \frac{(f_3, \phi_0)}{(\phi_0, \phi_0)}\,\phi_0(x) - \frac{(f_3, \phi_1)}{(\phi_1, \phi_1)}\,\phi_1(x) - \frac{(f_3, \phi_2)}{(\phi_2, \phi_2)}\,\phi_2(x).$$

(b) To show mutual orthogonality we compute (ϕ_0, ϕ_1), (ϕ_0, ϕ_2), and (ϕ_1, ϕ_2) using properties (i), (ii), and (iii) from this section in the text.

$$(\phi_0, \phi_1) = \left(\phi_0, f_1 - \frac{(f_1, \phi_0)}{(\phi_0, \phi_0)}\,\phi_0\right) = (\phi_0, f_1) - \frac{(f_1, \phi_0)}{(\phi_0, \phi_0)}\,(\phi_0, \phi_0) = (\phi_0, f_1) - (f_1, \phi_0) = 0$$

$$(\phi_0, \phi_2) = \left(\phi_0, f_2 - \frac{(f_2, \phi_0)}{(\phi_0, \phi_0)}\,\phi_0 - \frac{(f_2, \phi_1)}{(\phi_1, \phi_1)}\,\phi_1\right) = (\phi_0, f_2) - \frac{(f_2, \phi_0)}{(\phi_0, \phi_0)}\,(\phi_0, \phi_0) - \frac{(f_2, \phi_1)}{(\phi_1, \phi_1)}\,(\phi_0, \phi_1)$$
$$= (\phi_0, f_2) - (f_2, \phi_0) - 0 = 0$$

$$(\phi_1, \phi_2) = \left(\phi_1, f_2 - \frac{(f_2, \phi_0)}{(\phi_0, \phi_0)}\,\phi_0 - \frac{(f_2, \phi_1)}{(\phi_1, \phi_1)}\,\phi_1\right) = (\phi_1, f_2) - \frac{(f_2, \phi_0)}{(\phi_0, \phi_0)}\,(\phi_1, \phi_0) - \frac{(f_2, \phi_1)}{(\phi_1, \phi_1)}\,(\phi_1, \phi_1)$$
$$= (\phi_1, f_2) - 0 - (f_2, \phi_1) = 0.$$

23. (a) First we identify $f_0(x) = 1, f_1(x) = x, f_2(x) = x^2$, and $f_3(x) = x^3$. Then, we use the formulas from Problem 22. First, we have $\phi_0(x) = f_0(x) = 1$. Then

$$(f_1, \phi_0) = (x, 1) = \int_{-1}^{1} x\,dx = 0 \quad \text{and} \quad (\phi_0, \phi_0) = \int_{-1}^{1} 1\,dx = 2,$$

so

$$\phi_1(x) = f_1(x) - \frac{(f_1, \phi_0)}{(\phi_0, \phi_0)}\,\phi_0(x) = x - \frac{0}{2}\,(1) = 1.$$

Next

$$(f_2, \phi_0) = (x^2, 1) = \int_{-1}^{1} x^2 dx = \frac{2}{3}, \;\; (f_2, \phi_1) = (x^2, x) = \int_{-1}^{1} x^3 dx = 0, \;\; \text{and} \;\; (\phi_1, \phi_1) = \int_{-1}^{1} x^2 dx = \frac{2}{3},$$

so

$$\phi_2(x) = f_2(x) - \frac{(f_2, \phi_0)}{(\phi_0, \phi_0)}\,\phi_0(x) - \frac{(f_2, \phi_1)}{(\phi_1, \phi_1)}\,\phi_1(x) = x^2 - \frac{2/3}{2}\,(1) - \frac{0}{2}\,(x) = x^2 - \frac{1}{3}\,.$$

Finally,

$$(f_3, \phi_0) = (x^3, 1) = \int_{-1}^{1} x^3 dx = 0, \quad (f_3, \phi_1) = (x^3, x) = \int_{-1}^{1} x^4 dx = \frac{2}{5},$$

and

$$(f_3, \phi_2) = \left(x^3, x^2 - \frac{1}{3}\right) = \int_{-1}^{1} \left(x^5 - \frac{1}{3}\,x^3\right) dx = 0,$$

so

$$\phi_3(x) = f_3(x) - \frac{(f_3, \phi_0)}{(\phi_0, \phi_0)}\,\phi_0(x) - \frac{(f_3, \phi_1)}{(\phi_1, \phi_1)}\,\phi_1(x) - \frac{(f_3, \phi_2)}{(\phi_2, \phi_2)}\,\phi_2(x) = x^3 - 0 - \frac{2/5}{2/3}\,(x) - 0 = x^3 - \frac{3}{5}\,x.$$

(b) Recall from Section 5.3 that the first four Legendre polynomials are $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{3}{2}\,x^2 - \frac{1}{2}$, and $P_3(x) = \frac{5}{2}\,x^3 - \frac{3}{2}\,x$. We then see that $\phi_0(x) = P_0(x), \phi_1(x) = P_1(x), \phi_2(x) = x^2 - \frac{1}{3} = \frac{2}{3}(\frac{3}{2}\,x^2 - \frac{1}{2}) = \frac{2}{3}\,P_2(x)$, and $\phi_3(x) = x^3 - \frac{3}{5}\,x = \frac{2}{5}(\frac{5}{2}\,x^3 - \frac{3}{2}\,x) = \frac{2}{5}\,P_3(x)$.

24. (i): $(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx = \int_a^b f_2(x) f_1(x) dx = (f_2, f_1)$.

(ii): $(kf_1, f_2) = \int_a^b k f_1(x) f_2(x) dx = k \int_a^b f_1(x) f_2(x) dx = k(f_1, f_2)$.

(iii): If $f_1(x) = 0$ then $(f_1, f_1) = \int_a^b 0\,dx = 0$; if $f_1(x) \neq 0$ then $(f_1, f_1) = \int_a^b [f_1(x)]^2 dx > 0$ since $[f_1(x)]^2 > 0$.

(iv): $(f_1 + f_2, f_3) = \int_a^b [f_1(x) + f_2(x)] f_3(x) dx = \int_a^b [f_1(x) f_3(x) + f_2(x) f_3(x)] dx$
$= \int_a^b f_1(x) f_3(x) dx + \int_a^b f_2(x) f_3(x) dx = (f_1, f_3) + (f_2, f_3)$.

25. In R^3 the set $\{\mathbf{i}, \mathbf{j}\}$ is not complete since $\mathbf{k}$ is orthogonal to both $\mathbf{i}$ and $\mathbf{j}$. The set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is complete. To see this suppose that $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is orthogonal to $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Then

$$0 = (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}, \mathbf{i}) = a(\mathbf{i}, \mathbf{i}) + b(\mathbf{j}, \mathbf{i}) + c(\mathbf{k}, \mathbf{i}) = a(1) + b(0) + c(0) = a.$$

Similarly, $b = 0$ and $c = 0$. Thus, the only vector in R^3 orthogonal to $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ is $\mathbf{0}$, so $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is complete.

EXERCISES 12.2

Fourier Series

1. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} 1\,dx = 1$

$a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos\frac{n\pi}{\pi}x\,dx = \frac{1}{\pi}\int_0^{\pi}\cos nx\,dx = 0$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin\frac{n\pi}{\pi}x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin nx\,dx = \frac{1}{n\pi}(1-\cos n\pi) = \frac{1}{n\pi}[1-(-1)^n]$

$f(x) = \dfrac{1}{2} + \dfrac{1}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx$

2. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\,dx + \frac{1}{\pi}\int_0^{\pi} 2\,dx = 1$

$a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\cos nx\,dx + \frac{1}{\pi}\int_0^{\pi} 2\cos nx\,dx = 0$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\sin nx\,dx + \frac{1}{\pi}\int_0^{\pi} 2\sin nx\,dx = \frac{3}{n\pi}[1-(-1)^n]$

$f(x) = \dfrac{1}{2} + \dfrac{3}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx$

3. $a_0 = \displaystyle\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1\,dx + \int_0^{1} x\,dx = \frac{3}{2}$

$a_n = \displaystyle\int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_{-1}^{0}\cos n\pi x\,dx + \int_0^{1} x\cos n\pi x\,dx = \frac{1}{n^2\pi^2}[(-1)^n - 1]$

$b_n = \displaystyle\int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{-1}^{0}\sin n\pi x\,dx + \int_0^{1} x\sin n\pi x\,dx = -\frac{1}{n\pi}$

$f(x) = \dfrac{3}{4} + \displaystyle\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x - \frac{1}{n\pi}\sin n\pi x\right]$

4. $a_0 = \displaystyle\int_{-1}^{1} f(x)\,dx = \int_0^{1} x\,dx = \frac{1}{2}$

$a_n = \displaystyle\int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_0^{1} x\cos n\pi x\,dx = \frac{1}{n^2\pi^2}[(-1)^n - 1]$

$b_n = \displaystyle\int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_0^{1} x\sin n\pi x\,dx = \frac{(-1)^{n+1}}{n\pi}$

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2\pi^2} \cos n\pi x + \frac{(-1)^{n+1}}{n\pi} \sin n\pi x \right]$$

5. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{1}{3}\pi^2$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\cos nx\,dx = \frac{1}{\pi}\left(\frac{x^2}{\pi}\sin nx \,\Big|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin nx\,dx \right) = \frac{2(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x^2\sin nx\,dx = \frac{1}{\pi}\left(-\frac{x^2}{n}\cos nx \,\Big|_0^{\pi} + \frac{2}{n}\int_0^{\pi} x\cos nx\,dx \right) = \frac{\pi}{n}(-1)^{n+1} + \frac{2}{n^3\pi}[(-1)^n - 1]$$

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left[\frac{2(-1)^n}{n^2}\cos nx + \left(\frac{\pi}{n}(-1)^{n+1} + \frac{2[(-1)^n - 1]}{n^3\pi} \right) \sin nx \right]$$

6. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \pi^2\,dx + \frac{1}{\pi}\int_0^{\pi} \left(\pi^2 - x^2\right)\,dx = \frac{5}{3}\pi^2$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \pi^2\cos nx\,dx + \frac{1}{\pi}\int_0^{\pi} \left(\pi^2 - x^2\right)\cos nx\,dx$$

$$= \frac{1}{\pi}\left(\frac{\pi^2 - x^2}{n}\sin nx \,\Big|_0^{\pi} + \frac{2}{n}\int_0^{\pi} x\sin nx\,dx \right) = \frac{2}{n^2}(-1)^{n+1}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \pi^2\sin nx\,dx + \frac{1}{\pi}\int_0^{\pi} \left(\pi^2 - x^2\right)\sin nx\,dx$$

$$= \frac{\pi}{n}[(-1)^n - 1] + \frac{1}{\pi}\left(\frac{x^2 - \pi^2}{n}\cos nx \,\Big|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\cos nx\,dx \right) = \frac{\pi}{n}(-1)^n + \frac{2}{n^3\pi}[1 - (-1)^n]$$

$$f(x) = \frac{5\pi^2}{6} + \sum_{n=1}^{\infty} \left[\frac{2}{n^2}(-1)^{n+1}\cos nx + \left(\frac{\pi}{n}(-1)^n + \frac{2[1 - (-1)^n]}{n^3\pi} \right) \sin nx \right]$$

7. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + \pi)\,dx = 2\pi$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + \pi)\cos nx\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{n}(-1)^{n+1}$$

$$f(x) = \pi + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin nx$$

8. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (3 - 2x)\,dx = 6$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (3 - 2x)\cos nx\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (3 - 2x)\sin nx\,dx = \frac{4}{n}(-1)^n$$

$$f(x) = 3 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin nx$$

9. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = \frac{2}{\pi}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\,\cos nx\,dx = \frac{1}{2\pi}\int_0^{\pi} \Big(\sin(1+n)x + \sin(1-n)x\Big)\,dx$$

$$= \frac{1+(-1)^n}{\pi(1-n^2)} \quad \text{for } n = 2, 3, 4, \ldots$$

$$a_1 = \frac{1}{2\pi}\int_0^{\pi} \sin 2x\, dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{\pi} \sin x\, \sin nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} \Big(\cos(1-n)x - \cos(1+n)x\Big)\, dx = 0 \quad \text{for } n = 2, 3, 4, \ldots$$

$$b_1 = \frac{1}{2\pi}\int_0^{\pi} (1-\cos 2x)\, dx = \frac{1}{2}$$

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x + \sum_{n=2}^{\infty} \frac{1+(-1)^n}{\pi(1-n^2)}\cos nx$$

10. $$a_0 = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\, dx = \frac{2}{\pi}\int_0^{\pi/2} \cos x\, dx = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\cos 2nx\, dx = \frac{2}{\pi}\int_0^{\pi/2} \cos x\, \cos 2nx\, dx = \frac{1}{\pi}\int_0^{\pi/2} \Big(\cos(2n-1)x + \cos(2n+1)x\Big)\, dx$$

$$= \frac{2(-1)^{n+1}}{\pi(4n^2-1)}$$

$$b_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\sin 2nx\, dx = \frac{2}{\pi}\int_0^{\pi/2} \cos x\, \sin 2nx\, dx = \frac{1}{\pi}\int_0^{\pi/2} \Big(\sin(2n-1)x + \sin(2n+1)x\Big)\, dx$$

$$= \frac{4n}{\pi(4n^2-1)}$$

$$f(x) = \frac{1}{\pi} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^{n+1}}{\pi(4n^2-1)}\cos 2nx + \frac{4n}{\pi(4n^2-1)}\sin 2nx\right]$$

11. $$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\, dx = \frac{1}{2}\left(\int_{-1}^{0} -2\, dx + \int_0^{1} 1\, dx\right) = -\frac{1}{2}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\, dx = \frac{1}{2}\left(\int_{-1}^{0} (-2)\cos\frac{n\pi}{2}x\, dx + \int_0^{1} \cos\frac{n\pi}{2}x\, dx\right) = -\frac{1}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi}{2}x\, dx = \frac{1}{2}\left(\int_{-1}^{0} (-2)\sin\frac{n\pi}{2}x\, dx + \int_0^{1} \sin\frac{n\pi}{2}x\, dx\right) = \frac{3}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)$$

$$f(x) = -\frac{1}{4} + \sum_{n=1}^{\infty}\left[-\frac{1}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi}{2}x + \frac{3}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)\sin\frac{n\pi}{2}x\right]$$

12. $$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\, dx = \frac{1}{2}\left(\int_0^{1} x\, dx + \int_1^{2} 1\, dx\right) = \frac{3}{4}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\, dx = \frac{1}{2}\left(\int_0^{1} x\cos\frac{n\pi}{2}x\, dx + \int_1^{2} \cos\frac{n\pi}{2}x\, dx\right) = \frac{2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi}{2}x\, dx = \frac{1}{2}\left(\int_0^{1} x\sin\frac{n\pi}{2}x\, dx + \int_1^{2} \sin\frac{n\pi}{2}x\, dx\right)$$

$$= \frac{2}{n^2\pi^2}\left(\sin\frac{n\pi}{2} + \frac{n\pi}{2}(-1)^{n+1}\right)$$

$$f(x) = \frac{3}{8} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\cos\frac{n\pi}{2}x + \frac{2}{n^2\pi^2}\left(\sin\frac{n\pi}{2} + \frac{n\pi}{2}(-1)^{n+1}\right)\sin\frac{n\pi}{2}x\right]$$

13. $a_0 = \dfrac{1}{5}\displaystyle\int_{-5}^{5} f(x)\,dx = \frac{1}{5}\left(\int_{-5}^{0} 1\,dx + \int_{0}^{5}(1+x)\,dx\right) = \frac{9}{2}$

$a_n = \dfrac{1}{5}\displaystyle\int_{-5}^{5} f(x)\cos\frac{n\pi}{5}x\,dx = \frac{1}{5}\left(\int_{-5}^{0}\cos\frac{n\pi}{5}x\,dx + \int_{0}^{5}(1+x)\cos\frac{n\pi}{5}x\,dx\right) = \frac{5}{n^2\pi^2}[(-1)^n - 1]$

$b_n = \dfrac{1}{5}\displaystyle\int_{-5}^{5} f(x)\sin\frac{n\pi}{5}x\,dx = \frac{1}{5}\left(\int_{-5}^{0}\sin\frac{n\pi}{5}x\,dx + \int_{0}^{5}(1+x)\cos\frac{n\pi}{5}x\,dx\right) = \frac{5}{n\pi}(-1)^{n+1}$

$f(x) = \dfrac{9}{4} + \displaystyle\sum_{n=1}^{\infty}\left[\frac{5}{n^2\pi^2}[(-1)^n - 1]\cos\frac{n\pi}{5}x + \frac{5}{n\pi}(-1)^{n+1}\sin\frac{n\pi}{5}x\right]$

14. $a_0 = \dfrac{1}{2}\displaystyle\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left(\int_{-2}^{0}(2+x)\,dx + \int_{0}^{2} 2\,dx\right) = 3$

$a_n = \dfrac{1}{2}\displaystyle\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\,dx = \frac{1}{2}\left(\int_{-2}^{0}(2+x)\cos\frac{n\pi}{2}x\,dx + \int_{0}^{2} 2\cos\frac{n\pi}{2}x\,dx\right) = \frac{2}{n^2\pi^2}[1 - (-1)^n]$

$b_n = \dfrac{1}{2}\displaystyle\int_{-2}^{2} f(x)\sin\frac{n\pi}{2}x\,dx = \frac{1}{2}\left(\int_{-2}^{0}(2+x)\sin\frac{n\pi}{2}x\,dx + \int_{0}^{2} 2\sin\frac{n\pi}{2}x\,dx\right) = \frac{2}{n\pi}(-1)^{n+1}$

$f(x) = \dfrac{3}{2} + \displaystyle\sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi^2}[1 - (-1)^n]\cos\frac{n\pi}{2}x + \frac{2}{n\pi}(-1)^{n+1}\sin\frac{n\pi}{2}x\right]$

15. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi}(e^{\pi} - e^{-\pi})$

$a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{(-1)^n(e^{\pi} - e^{-\pi})}{\pi(1+n^2)}$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\sin nx\,dx = \frac{(-1)^n n(e^{-\pi} - e^{\pi})}{\pi(1+n^2)}$

$f(x) = \dfrac{e^{\pi} - e^{-\pi}}{2\pi} + \displaystyle\sum_{n=1}^{\infty}\left[\frac{(-1)^n(e^{\pi} - e^{-\pi})}{\pi(1+n^2)}\cos nx + \frac{(-1)^n n(e^{-\pi} - e^{\pi})}{\pi(1+n^2)}\sin nx\right]$

16. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi}(e^x - 1)\,dx = \frac{1}{\pi}(e^{\pi} - \pi - 1)$

$a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{0}^{\pi}(e^x - 1)\cos nx\,dx = \frac{[e^{\pi}(-1)^n - 1]}{\pi(1+n^2)}$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{0}^{\pi}(e^x - 1)\sin nx\,dx = \frac{1}{\pi}\left(\frac{ne^{\pi}(-1)^{n+1}}{1+n^2} + \frac{n}{1+n^2} + \frac{(-1)^n}{n} - \frac{1}{n}\right)$

$f(x) = \dfrac{e^{\pi} - \pi - 1}{2\pi} + \displaystyle\sum_{n=1}^{\infty}\left[\frac{e^{\pi}(-1)^n - 1}{\pi(1+n^2)}\cos nx + \left(\frac{n}{1+n^2}\left[e^{\pi}(-1)^{n+1} + 1\right] + \frac{(-1)^n - 1}{n}\right)\sin nx\right]$

17. The function in Problem 5 is discontinuous at $x = \pi$, so the corresponding Fourier series converges to $\pi^2/2$ at $x = \pi$. That is,

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{n^2}\cos n\pi + \left(\frac{\pi}{n}(-1)^{n+1} + \frac{2[(-1)^n - 1]}{n^3\pi}\right)\sin n\pi\right]$$

$$= \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}(-1)^n = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2}{n^2} = \frac{\pi^2}{6} + 2\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$

and

$$\frac{\pi^2}{6} = \frac{1}{2}\left(\frac{\pi^2}{2} - \frac{\pi^2}{6}\right) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots.$$

At $x = 0$ the series converges to 0 and

$$0 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} = \frac{\pi^2}{6} + 2\left(-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right)$$

so

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots.$$

18. From Problem 17

$$\frac{\pi^2}{8} = \frac{1}{2}\left(\frac{\pi^2}{6} + \frac{\pi^2}{12}\right) = \frac{1}{2}\left(2 + \frac{2}{3^2} + \frac{2}{5^2} + \cdots\right) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots.$$

19. The function in Problem 7 is continuous at $x = \pi/2$ so

$$\frac{3\pi}{2} = f\left(\frac{\pi}{2}\right) = \pi + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin\frac{n\pi}{2} = \pi + 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

and

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

20. The function in Problem 9 is continuous at $x = \pi/2$ so

$$1 = f\left(\frac{\pi}{2}\right) = \frac{1}{\pi} + \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1 + (-1)^n}{\pi(1 - n^2)} \cos\frac{n\pi}{2}$$

$$1 = \frac{1}{\pi} + \frac{1}{2} + \frac{2}{3\pi} - \frac{2}{3 \cdot 5\pi} + \frac{2}{5 \cdot 7\pi} - \cdots$$

and

$$\pi = 1 + \frac{\pi}{2} + \frac{2}{3} - \frac{2}{3 \cdot 5} + \frac{2}{5 \cdot 7} - \cdots$$

or

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \cdots.$$

21. Writing

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi}{p}x + \cdots + a_n \cos\frac{n\pi}{p}x + \cdots + b_1 \sin\frac{\pi}{p}x + \cdots + b_n \sin\frac{n\pi}{p}x + \cdots$$

we see that $f^2(x)$ consists exclusively of squared terms of the form

$$\frac{a_0^2}{4}, \qquad a_n^2 \cos^2\frac{n\pi}{p}x, \qquad b_n^2 \sin^2\frac{n\pi}{p}x$$

and cross-product terms, with $m \neq n$, of the form

$$a_0 a_n \cos\frac{n\pi}{p}x, \qquad a_0 b_n \sin\frac{n\pi}{p}x, \qquad 2a_m a_n \cos\frac{m\pi}{p}x \cos\frac{n\pi}{p}x,$$

$$2a_m b_n \cos\frac{m\pi}{p}x \sin\frac{n\pi}{p}x, \qquad 2b_m b_n \sin\frac{m\pi}{p}x \sin\frac{n\pi}{p}x.$$

The integral of each cross-product term taken over the interval $(-p, p)$ is zero by orthogonality. For the squared terms we have

$$\frac{a_0^2}{4}\int_{-p}^{p} dx = \frac{a_0^2 p}{2}, \qquad a_n^2 \int_{-p}^{p} \cos^2\frac{n\pi}{p}x\,dx = a_n^2 p, \qquad b_n^2 \int_{-p}^{p} \sin^2\frac{n\pi}{p}x\,dx = b_n^2 p.$$

Thus

$$RMS(f) = \sqrt{\frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)}\,.$$

EXERCISES 12.3

Fourier Cosine and Sine Series

1. Since $f(-x) = \sin(-3x) = -\sin 3x = -f(x)$, $f(x)$ is an odd function.

2. Since $f(-x) = -x\cos(-x) = -x\cos x = -f(x)$, $f(x)$ is an odd function.

3. Since $f(-x) = (-x)^2 - x = x^2 - x$, $f(x)$ is neither even nor odd.

4. Since $f(-x) = (-x)^3 + 4x = -(x^3 - 4x) = -f(x)$, $f(x)$ is an odd function.

5. Since $f(-x) = e^{|-x|} = e^{|x|} = f(x)$, $f(x)$ is an even function.

6. Since $f(-x) = e^{-x} - e^{x} = -f(x)$, $f(x)$ is an odd function.

7. For $0 < x < 1$, $f(-x) = (-x)^2 = x^2 = -f(x)$, $f(x)$ is an odd function.

8. For $0 \le x < 2$, $f(-x) = -x + 5 = f(x)$, $f(x)$ is an even function.

9. Since $f(x)$ is not defined for $x < 0$, it is neither even nor odd.

10. Since $f(-x) = \left|(-x)^5\right| = \left|x^5\right| = f(x)$, $f(x)$ is an even function.

11. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin nx\, dx = \frac{2}{n\pi}\left[1 - (-1)^n\right].$$

Thus

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1 - (-1)^n\right]\sin nx.$$

12. Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \int_1^2 1\, dx = 1$$

$$a_n = \int_1^2 \cos\frac{n\pi}{2}x\, dx = -\frac{2}{n\pi}\sin\frac{n\pi}{2}.$$

Thus

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-2}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi}{2}x.$$

13. Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\, dx = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\, dx = \frac{2}{n^2\pi}[(-1)^n - 1].$$

Thus

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}[(-1)^n - 1]\cos nx.$$

14. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = \frac{2}{\pi}\int_0^\pi x\sin nx\,dx = \frac{2}{n}(-1)^{n+1}.$$

Thus

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx.$$

15. Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = 2\int_0^1 x^2\,dx = \frac{2}{3}$$

$$a_n = 2\int_0^1 x^2\cos n\pi x\,dx = 2\left(\frac{x^2}{n\pi}\sin n\pi x\,\Big|_0^1 - \frac{2}{n\pi}\int_0^1 x\sin n\pi x\,dx\right) = \frac{4}{n^2\pi^2}(-1)^n.$$

Thus

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}(-1)^n\cos n\pi x.$$

16. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = 2\int_0^1 x^2\sin n\pi x\,dx = 2\left(-\frac{x^2}{n\pi}\cos n\pi x\,\Big|_0^1 + \frac{2}{n\pi}\int_0^1 x\cos n\pi x\,dx\right) = \frac{2(-1)^{n+1}}{n\pi} + \frac{4}{n^3\pi^3}[(-1)^n - 1].$$

Thus

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{n\pi} + \frac{4}{n^3\pi^3}[(-1)^n - 1]\right)\sin n\pi x.$$

17. Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \frac{2}{\pi}\int_0^\pi (\pi^2 - x^2)\,dx = \frac{4}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^\pi (\pi^2 - x^2)\cos nx\,dx = \frac{2}{\pi}\left(\frac{\pi^2 - x^2}{n}\sin nx\,\Big|_0^\pi + \frac{2}{n}\int_0^\pi x\sin nx\,dx\right) = \frac{4}{n^2}(-1)^{n+1}.$$

Thus

$$f(x) = \frac{2}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^{n+1}\cos nx\,dx.$$

18. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = \frac{2}{\pi}\int_0^\pi x^3\sin nx\,dx = \frac{2}{\pi}\left(-\frac{x^3}{n}\cos nx\,\Big|_0^\pi + \frac{3}{n}\int_0^\pi x^2\cos nx\,dx\right) = \frac{2\pi^2}{n}(-1)^{n+1} - \frac{12}{n^2\pi}\int_0^\pi x\sin nx\,dx$$

$$= \frac{2\pi^2}{n}(-1)^{n+1} - \frac{12}{n^2\pi}\left(-\frac{x}{n}\cos nx\,\Big|_0^\pi + \frac{1}{n}\int_0^\pi \cos nx\,dx\right) = \frac{2\pi^2}{n}(-1)^{n+1} + \frac{12}{n^3}(-1)^n.$$

Thus

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2\pi^2}{n}(-1)^{n+1} + \frac{12}{n^3}(-1)^n\right)\sin nx.$$

19. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = \frac{2}{\pi}\int_0^\pi (x+1)\sin nx\,dx = \frac{2(\pi+1)}{n\pi}(-1)^{n+1} + \frac{2}{n\pi}.$$

Thus

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{2(\pi+1)}{n\pi}(-1)^{n+1} + \frac{2}{n\pi} \right) \sin nx.$$

20. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = 2\int_0^1 (x-1)\sin n\pi x\,dx = 2\left[\int_0^1 x\sin n\pi x\,dx - \int_0^1 \sin n\pi x\,dx\right]$$

$$= 2\left[\frac{1}{n^2\pi^2}\sin n\pi x - \frac{x}{n\pi}\cos n\pi x + \frac{1}{n\pi}\cos n\pi x\right]_0^1 = -\frac{2}{n\pi}.$$

Thus

$$f(x) = -\sum_{n=1}^{\infty} \frac{2}{n\pi}\sin n\pi x.$$

21. Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \int_0^1 x\,dx + \int_1^2 1\,dx = \frac{3}{2}$$

$$a_n = \int_0^1 x\cos\frac{n\pi}{2}x\,dx + \int_1^2 \cos\frac{n\pi}{2}x\,dx = \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right).$$

Thus

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\cos\frac{n\pi}{2}x.$$

22. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin\frac{n}{2}x\,dx + \int_{\pi}^{2\pi} \pi\sin\frac{n}{2}x\,dx = \frac{4}{n^2\pi}\sin\frac{n\pi}{2} + \frac{2}{n}(-1)^{n+1}.$$

Thus

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{4}{n^2\pi}\sin\frac{n\pi}{2} + \frac{2}{n}(-1)^{n+1} \right) \sin\frac{n}{2}x.$$

23. Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x\,dx = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x\,\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} \Big(\sin(1+n)x + \sin(1-n)x\Big)\,dx$$

$$= \frac{2}{\pi(1-n^2)}(1+(-1)^n) \quad \text{for } n = 2, 3, 4, \ldots$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin 2x\,dx = 0.$$

Thus

$$f(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{2[1+(-1)^n]}{\pi(1-n^2)}\cos nx.$$

24. Since $f(x)$ is an even function, we expand in a cosine series. [See the solution of Problem 10 in Exercises 12.2 for the computation of the integrals.]

$$a_0 = \frac{2}{\pi/2}\int_0^{\pi/2} \cos x\,dx = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi/2}\int_0^{\pi/2} \cos x \cos\frac{n\pi}{\pi/2}x\,dx = \frac{4(-1)^{n+1}}{\pi\,(4n^2-1)}$$

Thus

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{\pi\,(4n^2-1)} \cos 2nx.$$

25. $a_0 = 2\displaystyle\int_0^{1/2} 1\,dx = 1$

$a_n = 2\displaystyle\int_0^{1/2} 1\cdot\cos n\pi x\,dx = \frac{2}{n\pi}\sin\frac{n\pi}{2}$

$b_n = 2\displaystyle\int_0^{1/2} 1\cdot\sin n\pi x\,dx = \frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)$

$f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n\pi}{2}\cos n\pi x$

$f(x) = \displaystyle\sum_{n=1}^{\infty}\frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)\sin n\pi x$

26. $a_0 = 2\displaystyle\int_{1/2}^{1} 1\,dx = 1$

$a_n = 2\displaystyle\int_{1/2}^{1} 1\cdot\cos n\pi x\,dx = -\frac{2}{n\pi}\sin\frac{n\pi}{2}$

$b_n = 2\displaystyle\int_{1/2}^{1} 1\cdot\sin n\pi x\,dx = \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} + (-1)^{n+1}\right)$

$f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\sin\frac{n\pi}{2}\right)\cos n\pi x$

$f(x) = \displaystyle\sum_{n=1}^{\infty}\frac{2}{n\pi}\left(\cos\frac{n\pi}{2} + (-1)^{n+1}\right)\sin n\pi x$

27. $a_0 = \dfrac{4}{\pi}\displaystyle\int_0^{\pi/2}\cos x\,dx = \frac{4}{\pi}$

$a_n = \dfrac{4}{\pi}\displaystyle\int_0^{\pi/2}\cos x\,\cos 2nx\,dx = \frac{2}{\pi}\int_0^{\pi/2}[\cos(2n+1)x + \cos(2n-1)x]\,dx = \frac{4(-1)^n}{\pi(1-4n^2)}$

$b_n = \dfrac{4}{\pi}\displaystyle\int_0^{\pi/2}\cos x\,\sin 2nx\,dx = \frac{2}{\pi}\int_0^{\pi/2}[\sin(2n+1)x + \sin(2n-1)x]\,dx = \frac{8n}{\pi(4n^2-1)}$

$f(x) = \dfrac{2}{\pi} + \displaystyle\sum_{n=1}^{\infty}\frac{4(-1)^n}{\pi(1-4n^2)}\cos 2nx$

$f(x) = \displaystyle\sum_{n=1}^{\infty}\frac{8n}{\pi(4n^2-1)}\sin 2nx$

28. $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \sin x\,dx = \dfrac{4}{\pi}$

$a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \sin x\,\cos nx\,dx = \dfrac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\,dx = \dfrac{2[(-1)^n+1]}{\pi(1-n^2)}$ for $n = 2, 3, 4, \ldots$

$b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \sin x\,\sin nx\,dx = \dfrac{1}{\pi}\int_0^{\pi} [\cos(n-1)x - \cos(n+1)x]\,dx = 0$ for $n = 2, 3, 4, \ldots$

$a_1 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \sin 2x\,dx = 0$

$b_1 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \sin^2 x\,dx = 1$

$f(x) = \sin x$

$f(x) = \dfrac{2}{\pi} + \dfrac{2}{\pi}\displaystyle\sum_{n=2}^{\infty} \dfrac{(-1)^n+1}{1-n^2}\,\cos nx$

29. $a_0 = \dfrac{2}{\pi}\left(\displaystyle\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right) = \dfrac{\pi}{2}$

$a_n = \dfrac{2}{\pi}\left(\displaystyle\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx\right) = \dfrac{2}{n^2\pi}\left(2\cos\dfrac{n\pi}{2} + (-1)^{n+1} - 1\right)$

$b_n = \dfrac{2}{\pi}\left(\displaystyle\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\,dx\right) = \dfrac{4}{n^2\pi}\sin\dfrac{n\pi}{2}$

$f(x) = \dfrac{\pi}{4} + \displaystyle\sum_{n=1}^{\infty} \dfrac{2}{n^2\pi}\left(2\cos\dfrac{n\pi}{2} + (-1)^{n+1} - 1\right)\cos nx$

$f(x) = \displaystyle\sum_{n=1}^{\infty} \dfrac{4}{n^2\pi}\sin\dfrac{n\pi}{2}\,\sin nx$

30. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{\pi}^{2\pi} (x - \pi)\,dx = \dfrac{\pi}{2}$

$a_n = \dfrac{1}{\pi}\displaystyle\int_{\pi}^{2\pi} (x - \pi)\cos\dfrac{n}{2}x\,dx = \dfrac{4}{n^2\pi}\left((-1)^n - \cos\dfrac{n\pi}{2}\right)$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{\pi}^{2\pi} (x - \pi)\sin\dfrac{n}{2}x\,dx = \dfrac{2}{n}(-1)^{n+1} - \dfrac{4}{n^2\pi}\sin\dfrac{n\pi}{2}$

$f(x) = \dfrac{\pi}{4} + \displaystyle\sum_{n=1}^{\infty} \dfrac{4}{n^2\pi}\left((-1)^n - \cos\dfrac{n\pi}{2}\right)\cos\dfrac{n}{2}x$

$f(x) = \displaystyle\sum_{n=1}^{\infty} \left(\dfrac{2}{n}(-1)^{n+1} - \dfrac{4}{n^2\pi}\sin\dfrac{n\pi}{2}\right)\sin\dfrac{n}{2}x$

31. $a_0 = \displaystyle\int_0^1 x\,dx + \int_1^2 1\,dx = \dfrac{3}{2}$

$a_n = \displaystyle\int_0^1 x\cos\dfrac{n\pi}{2}x\,dx = \dfrac{4}{n^2\pi^2}\left(\cos\dfrac{n\pi}{2} - 1\right)$

$b_n = \displaystyle\int_0^1 x\sin\dfrac{n\pi}{2}x\,dx + \int_1^2 1\cdot\sin\dfrac{n\pi}{2}x\,dx = \dfrac{4}{n^2\pi^2}\sin\dfrac{n\pi}{2} + \dfrac{2}{n\pi}(-1)^{n+1}$

$f(x) = \dfrac{3}{4} + \displaystyle\sum_{n=1}^{\infty} \dfrac{4}{n^2\pi^2}\left(\cos\dfrac{n\pi}{2} - 1\right)\cos\dfrac{n\pi}{2}x$

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{2}{n\pi}(-1)^{n+1} \right) \sin \frac{n\pi}{2} x$$

32. $a_0 = \displaystyle\int_0^1 1\,dx + \int_1^2 (2-x)\,dx = \frac{3}{2}$

$$a_n = \int_0^1 1 \cdot \cos \frac{n\pi}{2} x\,dx + \int_1^2 (2-x) \cos \frac{n\pi}{2} x\,dx = \frac{4}{n^2\pi^2} \left(\cos \frac{n\pi}{2} + (-1)^{n+1} \right)$$

$$b_n = \int_0^1 1 \cdot \sin \frac{n\pi}{2} x\,dx + \int_1^2 (2-x) \sin \frac{n\pi}{2} x\,dx = \frac{2}{n\pi} + \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2}$$

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \left(\cos \frac{n\pi}{2} + (-1)^{n+1} \right) \cos \frac{n\pi}{2} x$$

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{2}{n\pi} + \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} \right) \sin \frac{n\pi}{2} x$$

33. $a_0 = 2\displaystyle\int_0^1 (x^2 + x)\,dx = \frac{5}{3}$

$$a_n = 2\int_0^1 (x^2+x) \cos n\pi x\,dx = \frac{2(x^2+x)}{n\pi} \sin n\pi x \,\Big|_0^1 - \frac{2}{n\pi} \int_0^1 (2x+1) \sin n\pi x\,dx = \frac{2}{n^2\pi^2}[3(-1)^n - 1]$$

$$b_n = 2\int_0^1 (x^2+x) \sin n\pi x\,dx = -\frac{2(x^2+x)}{n\pi} \cos n\pi x \,\Big|_0^1 + \frac{2}{n\pi} \int_0^1 (2x+1) \cos n\pi x\,dx$$

$$= \frac{4}{n\pi}(-1)^{n+1} + \frac{4}{n^3\pi^3}[(-1)^n - 1]$$

$$f(x) = \frac{5}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}[3(-1)^n - 1] \cos n\pi x$$

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{4}{n\pi}(-1)^{n+1} + \frac{4}{n^3\pi^3}[(-1)^n - 1] \right) \sin n\pi x$$

34. $a_0 = \displaystyle\int_0^2 (2x - x^2)\,dx = \frac{4}{3}$

$$a_n = \int_0^2 (2x - x^2) \cos \frac{n\pi}{2} x\,dx = \frac{8}{n^2\pi^2}[(-1)^{n+1} - 1]$$

$$b_n = \int_0^2 (2x - x^2) \sin \frac{n\pi}{2} x\,dx = \frac{16}{n^3\pi^3}[1 - (-1)^n]$$

$$f(x) = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2}[(-1)^{n+1} - 1] \cos \frac{n\pi}{2} x$$

$$f(x) = \sum_{n=1}^{\infty} \frac{16}{n^3\pi^3}[1 - (-1)^n] \sin \frac{n\pi}{2} x$$

35. $a_0 = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} x^2\,dx = \frac{8}{3}\pi^2$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx\,dx = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} x^2 \sin nx\,dx = -\frac{4\pi}{n}$$

$$f(x) = \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

36. $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} x\,dx = \pi$

$a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} x\cos 2nx\,dx = 0$

$b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} x\sin 2nx\,dx = -\dfrac{1}{n}$

$f(x) = \dfrac{\pi}{2} - \displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\sin 2nx$

37. $a_0 = 2\displaystyle\int_0^{1}(x+1)\,dx = 3$

$a_n = 2\displaystyle\int_0^{1}(x+1)\cos 2n\pi x\,dx = 0$

$b_n = 2\displaystyle\int_0^{1}(x+1)\sin 2n\pi x\,dx = -\dfrac{1}{n\pi}$

$f(x) = \dfrac{3}{2} - \displaystyle\sum_{n=1}^{\infty}\frac{1}{n\pi}\sin 2n\pi x$

38. $a_0 = \dfrac{2}{2}\displaystyle\int_0^{2}(2-x)\,dx = 2$

$a_n = \dfrac{2}{2}\displaystyle\int_0^{2}(2-x)\cos n\pi x\,dx = 0$

$b_n = \dfrac{2}{2}\displaystyle\int_0^{2}(2-x)\sin n\pi x\,dx = \dfrac{2}{n\pi}$

$f(x) = 1 + \displaystyle\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin n\pi x$

39. We have

$$b_n = \frac{2}{\pi}\int_0^{\pi} 5\sin nt\,dt = \frac{10}{n\pi}[1-(-1)^n]$$

so that

$$f(t) = \sum_{n=1}^{\infty}\frac{10[1-(-1)^n]}{n\pi}\sin nt.$$

Substituting the assumption $x_p(t) = \sum_{n=1}^{\infty} B_n \sin nt$ into the differential equation then gives

$$x_p'' + 10x_p = \sum_{n=1}^{\infty} B_n(10-n^2)\sin nt = \sum_{n=1}^{\infty}\frac{10[1-(-1)^n]}{n\pi}\sin nt$$

and so $B_n = 10[1-(-1)^n]/n\pi(10-n^2)$. Thus

$$x_p(t) = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n(10-n^2)}\sin nt.$$

40. We have

$$b_n = \frac{2}{\pi}\int_0^{1}(1-t)\sin n\pi t\,dt = \frac{2}{n\pi}$$

so that

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin n\pi t.$$

Substituting the assumption $x_p(t) = \sum_{n=1}^{\infty} B_n \sin n\pi t$ into the differential equation then gives

$$x_p'' + 10x_p = \sum_{n=1}^{\infty} B_n(10 - n^2\pi^2) \sin n\pi t = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin n\pi t$$

and so $B_n = 2/n\pi(10 - n^2\pi^2)$. Thus

$$x_p(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n(10 - n^2\pi^2)} \sin n\pi t.$$

41. We have

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (2\pi t - t^2)\, dt = \frac{4}{3}\pi^2$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} (2\pi t - t^2) \cos nt\, dt = -\frac{4}{n^2}$$

so that

$$f(t) = \frac{2\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nt.$$

Substituting the assumption

$$x_p(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nt$$

into the differential equation then gives

$$\frac{1}{4} x_p'' + 12x_p = 6A_0 + \sum_{n=1}^{\infty} A_n \left(-\frac{1}{4}n^2 + 12\right) \cos nt = \frac{2\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nt$$

and $A_0 = \pi^2/9$, $A_n = 16/n^2(n^2 - 48)$. Thus

$$x_p(t) = \frac{\pi^2}{18} + 16 \sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - 48)} \cos nt.$$

42. We have

$$a_0 = \frac{2}{1/2} \int_0^{1/2} t\, dt = \frac{1}{2}$$

$$a_n = \frac{2}{1/2} \int_0^{1/2} t \cos 2n\pi t\, dt = \frac{1}{n^2\pi^2} [(-1)^n - 1]$$

so that

$$f(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi^2} \cos 2n\pi t.$$

Substituting the assumption

$$x_p(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos 2n\pi t$$

into the differential equation then gives

$$\frac{1}{4} x_p'' + 12x_p = 6A_0 + \sum_{n=1}^{\infty} A_n(12 - n^2\pi^2) \cos 2n\pi t = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi^2} \cos 2n\pi t$$

and $A_0 = 1/24$, $A_n = [(-1)^n - 1]/n^2\pi^2(12 - n^2\pi^2)$. Thus

$$x_p(t) = \frac{1}{48} + \frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2(12 - n^2\pi^2)} \cos 2n\pi t.$$

43. (a) The general solution is $x(t) = c_1 \cos\sqrt{10}t + c_2 \sin\sqrt{10}t + x_p(t)$, where

$$x_p(t) = \frac{10}{\pi}\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n(10 - n^2)} \sin nt.$$

The initial condition $x(0) = 0$ implies $c_1 + x_p(0) = 0$. Since $x_p(0) = 0$, we have $c_1 = 0$ and $x(t) = c_2 \sin\sqrt{10}t + x_p(t)$. Then $x'(t) = c_2\sqrt{10}\cos\sqrt{10}t + x_p'(t)$ and $x'(0) = 0$ implies

$$c_2\sqrt{10} + \frac{10}{\pi}\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{10 - n^2} \cos 0 = 0.$$

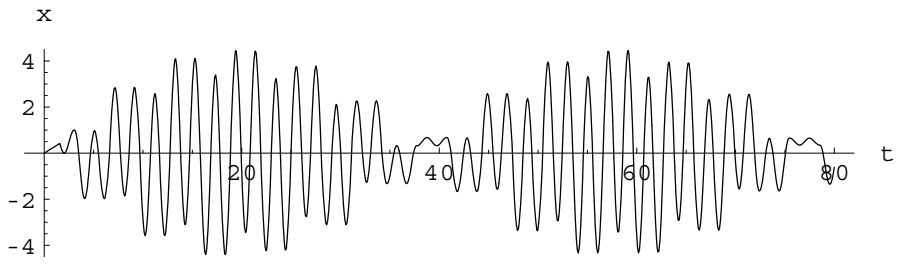
Thus

$$c_2 = -\frac{\sqrt{10}}{\pi}\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{10 - n^2}$$

and

$$x(t) = \frac{10}{\pi}\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{10 - n^2}\left[\frac{1}{n}\sin nt - \frac{1}{\sqrt{10}}\sin\sqrt{10}t\right].$$

(b) The graph is plotted using eight nonzero terms in the series expansion of $x(t)$.



44. (a) The general solution is $x(t) = c_1 \cos 4\sqrt{3}t + c_2 \sin 4\sqrt{3}t + x_p(t)$, where

$$x_p(t) = \frac{\pi^2}{18} + 16\sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - 48)} \cos nt.$$

The initial condition $x(0) = 0$ implies $c_1 + x_p(0) = 1$ or

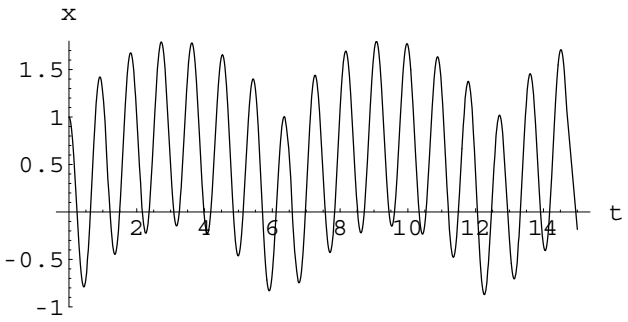
$$c_1 = 1 - x_p(0) = 1 - \frac{\pi^2}{18} - 16\sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - 48)}.$$

Now $x'(t) = -4\sqrt{3}c_1 \sin 4\sqrt{3}t + 4\sqrt{3}c_2 \cos 4\sqrt{3}t + x_p'(t)$, so $x'(0) = 0$ implies $4\sqrt{3}c_2 + x_p'(0) = 0$. Since $x_p'(0) = 0$, we have $c_2 = 0$ and

$$x(t) = \left(1 - \frac{\pi^2}{18} - 16\sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - 48)}\right)\cos 4\sqrt{3}t + \frac{\pi^2}{18} + 16\sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - 48)} \cos nt$$

$$= \frac{\pi^2}{18} + \left(1 - \frac{\pi^2}{18}\right)\cos 4\sqrt{3}t + 16\sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - 48)}\left[\cos nt - \cos 4\sqrt{3}t\right].$$

(b) The graph is plotted using five nonzero terms in the series expansion of $x(t)$.



45. (a) We have

$$b_n = \frac{2}{L}\int_0^L \frac{w_0 x}{L}\,\sin\frac{n\pi}{L}\,x\,dx = \frac{2w_0}{n\pi}\,(-1)^{n+1}$$

so that

$$w(x) = \sum_{n=1}^{\infty} \frac{2w_0}{n\pi}\,(-1)^{n+1}\sin\frac{n\pi}{L}\,x.$$

(b) If we assume $y_p(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/L)$ then

$$y_p^{(4)} = \sum_{n=1}^{\infty} \frac{n^4\pi^4}{L^4}\,B_n \sin\frac{n\pi}{L}\,x$$

and so the differential equation $EIy_p^{(4)} = w(x)$ gives

$$B_n = \frac{2w_0(-1)^{n+1}L^4}{EIn^5\pi^5}.$$

Thus

$$y_p(x) = \frac{2w_0L^4}{EI\pi^5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}\,\sin\frac{n\pi}{L}\,x.$$

46. We have

$$b_n = \frac{2}{L}\int_{L/3}^{2L/3} w_0 \sin\frac{n\pi}{L}\,x\,dx = \frac{2w_0}{n\pi}\left(\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right)$$

so that

$$w(x) = \sum_{n=1}^{\infty}\frac{2w_0}{n\pi}\left(\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right)\sin\frac{n\pi}{L}\,x.$$

If we assume $y_p(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/L)$ then

$$y_p^{(4)}(x) = \sum_{n=1}^{\infty}\frac{n^4\pi^4}{L^4}\,B_n\sin\frac{n\pi}{L}\,x$$

and so the differential equation $EIy_p^{(4)}(x) = w(x)$ gives

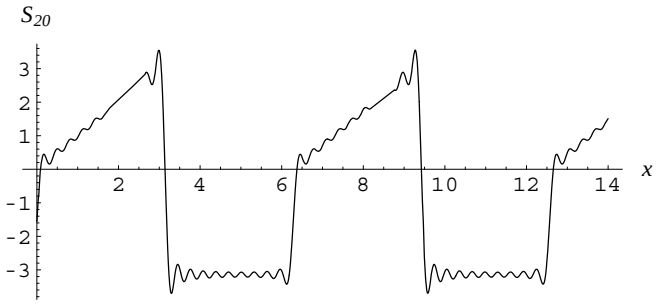
$$B_n = 2w_0L^4\frac{\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}}{EIn^5\pi^5}.$$

Thus

$$y_p(x) = \frac{2w_0L^4}{EI\pi^5}\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}}{n^5}\,\sin\frac{n\pi}{L}\,x.$$

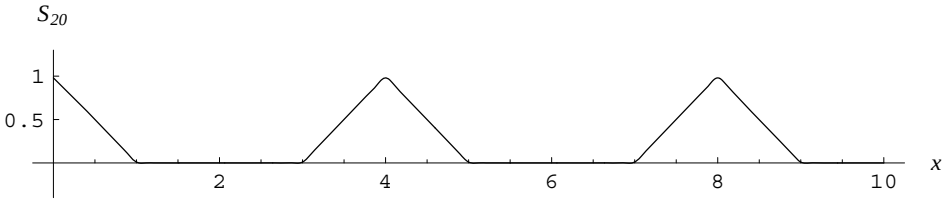
47. The graph is obtained by summing the series from $n = 1$ to 20. It appears that

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ -\pi, & \pi < x < 2\pi. \end{cases}$$



48. The graph is obtained by summing the series from $n = 1$ to 10. It appears that

$$f(x) = \begin{cases} 1 - x, & 0 < x < 1 \\ 0, & 1 < x < 2. \end{cases}$$



49. The function in Problem 47 is not unique; it could also be defined as

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ 1, & x = \pi \\ -\pi, & \pi < x < 2\pi. \end{cases}$$

The function in Problem 48 is not unique; it could also be defined as

$$f(x) = \begin{cases} 0, & -2 < x < -1 \\ x + 1, & -1 < x < 0 \\ -x + 1, & 0 < x < 1 \\ 0, & 1 < x < 2. \end{cases}$$

50. The cosine series converges to an even extension of the function on the interval $(-\pi, 0)$. Since the even extension of $f(x)$ is $f(-x)$, in this case $f(-x) = e^{-x}$ on $(-\pi, 0)$.

51. No, it is not a full Fourier series. A full Fourier series of $f(x) = e^x, 0 < x < \pi$, would converge to the π-periodic extension of f. The cosine and sine series converge to a 2π-periodic extension (even and odd, respectively). The average of the two series converges to a 2π-periodic extension of

$$f(x) = \begin{cases} e^x, & 0 < x < \pi \\ 0, & -\pi < x < 0. \end{cases}$$

52. (a) If f and g are even and $h(x) = f(x)g(x)$ then

$$h(-x) = f(-x)g(-x) = f(x)g(x) = h(x)$$

and h is even.

(c) If f is even and g is odd and $h(x) = f(x)g(x)$ then

$$h(-x) = f(-x)g(-x) = f(x)[-g(x)] = -h(x)$$

and h is odd.

(d) Let $h(x) = f(x) \pm g(x)$ where f and g are even. Then

$$h(-x) = f(-x) \pm g(-x) = f(x) \pm g(x) = h(x),$$

and so h is an even function.

(f) If f is even then

$$\int_{-a}^{a} f(x)\,dx = -\int_{a}^{0} f(-u)\,du + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

(g) If f is odd then

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= -\int_{-a}^{0} f(-x)\,dx + \int_{0}^{a} f(x)\,dx = \int_{a}^{0} f(u)\,du + \int_{0}^{a} f(x)\,dx \\ &= -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 0.\end{aligned}$$

EXERCISES 12.4

Complex Fourier Series

In this section we make use of the following identities due to Euler's formula:

$$e^{in\pi} = e^{-in\pi} = (-1)^n, \qquad e^{-2in\pi} = 1, \qquad e^{-in\pi/2} = (-i)^n.$$

1. Identifying $p = 2$ we have

$$\begin{aligned}c_n &= \frac{1}{4}\int_{-2}^{2} f(x)e^{-in\pi x/2}dx = \frac{1}{4}\left[\int_{-2}^{0}(-1)e^{-in\pi x/2}dx + \int_{0}^{2} e^{-in\pi x/2}dx\right] \\ &= \frac{i}{2n\pi}\left[-1 + e^{in\pi} + e^{-in\pi} - 1\right] = \frac{i}{2n\pi}\left[-1 + (-1)^n + (-1)^n - 1\right] = \frac{1-(-1)^n}{n\pi i}\end{aligned}$$

and

$$c_0 = \frac{1}{4}\int_{-2}^{2} f(x)dx = 0.$$

Thus

$$f(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1-(-1)^n}{in\pi} e^{in\pi x/2}.$$

2. Identifying $2p = 2$ or $p = 1$ we have

$$c_n = \frac{1}{2}\int_0^2 f(x)e^{-in\pi x}dx = \frac{1}{2}\int_1^2 e^{-in\pi x}dx = -\frac{1}{2in\pi}e^{-in\pi x}\Big|_1^2$$
$$= -\frac{1}{2in\pi}\left(e^{-2in\pi} - e^{-in\pi}\right) = -\frac{1}{2in\pi}[1-(-1)^n] = \frac{i}{2n\pi}[1-(-1)^n]$$

and

$$c_0 = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}\int_1^2 dx = \frac{1}{2}\,.$$

Thus

$$f(x) = \frac{1}{2} + \frac{i}{2\pi}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1-(-1)^n}{n}e^{in\pi x}.$$

3. Identifying $p = 1/2$ we have

$$c_n = \int_{-1/2}^{1/2} f(x)e^{-2in\pi x}dx = \int_0^{1/4} e^{-2in\pi x}dx = -\frac{1}{2in\pi}e^{-2in\pi x}\Big|_0^{1/4}$$
$$= -\frac{1}{2in\pi}\left[e^{-in\pi/2} - 1\right] = -\frac{1}{2in\pi}[(-i)^n - 1] = \frac{i}{2n\pi}[(-i)^n - 1]$$

and

$$c_0 = \int_0^{1/4} dx = \frac{1}{4}\,.$$

Thus

$$f(x) = \frac{1}{4} + \frac{i}{2\pi}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-i)^n - 1}{n}e^{2in\pi x}.$$

4. Identifying $p = \pi$ we have

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx/\pi}dx = \frac{1}{2\pi}\int_0^{\pi} xe^{-inx/\pi}dx$$
$$= \frac{1}{2}\left(\frac{\pi}{n^2} + \frac{ix}{n}\right)e^{-inx/\pi}\Big|_0^{\pi} = \frac{\pi(1+in)}{2n^2}e^{-in} - \frac{\pi}{2n^2}$$

and

$$c_0 = \frac{1}{2\pi}\int_0^{\pi} x\,dx = \frac{\pi}{4}\,.$$

Thus

$$f(x) = \frac{\pi}{4} + \frac{\pi}{2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{n^2}\Big[(1+in)e^{-in} - 1\Big]e^{inx}.$$

5. Identifying $2p = 2\pi$ or $p = \pi$ we have

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_0^{2\pi} xe^{-inx}dx$$
$$= \frac{1}{2\pi}\left(\frac{1}{n^2} + \frac{ix}{n}\right)e^{-inx}\Big|_0^{2\pi} = \frac{1+2in\pi}{2n^2\pi} - \frac{1}{2n^2\pi} = \frac{i}{n}$$

and

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \pi.$$

Thus

$$f(x) = \pi + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i}{n} e^{inx}.$$

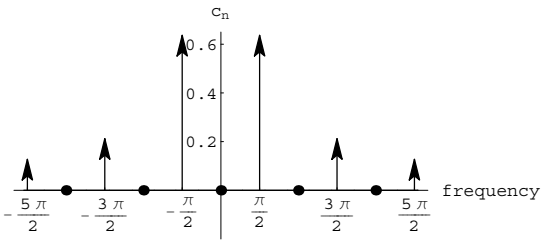
6. Identifying $p = 1$ we have

$$\begin{aligned} c_n &= \frac{1}{2}\int_{-1}^{1} f(x)e^{-in\pi x}dx = \frac{1}{2}\left[\int_{-1}^{0} e^x e^{-in\pi x}dx + \int_0^1 e^{-x}e^{-in\pi x}dx\right] \\ &= \frac{1}{2}\left[-\frac{1}{1-in\pi}e^{(1-in\pi)x}\Big|_{-1}^{0} - \frac{1}{1+in\pi}e^{-(1+in\pi)x}\Big|_0^1\right] \\ &= \frac{e-(-1)^n}{e(1-in\pi)} + \frac{1-e^{-1}(-1)^n}{1+in\pi} = \frac{2[e-(-1)^n]}{e(1+n^2\pi^2)}. \end{aligned}$$

Thus

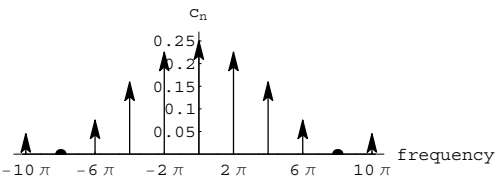
$$f(x) = \sum_{n=-\infty}^{\infty} \frac{2[e-(-1)^n]}{e(1+n^2\pi^2)} e^{in\pi x}.$$

7. The fundamental period is $T = 4$, so $\omega = 2\pi/4 = \pi/2$ and the values of $n\omega$ are $0, \pm\pi/2, \pm\pi, \pm 3\pi/2, \ldots$. From Problem 1, $c_0 = 0$ and $|c_n| = (1-(-1)^n)/n\pi$. The table shows some values of n with corresponding values of $|c_n|$. The graph is a portion of the frequency spectrum.



n	-5	-4	-3	-2	-1	0	1	2	3	4	5
c_n	0.1273	0.0000	0.2122	0.0000	0.6366	0.0000	0.6366	0.0000	0.2122	0.0000	0.1273

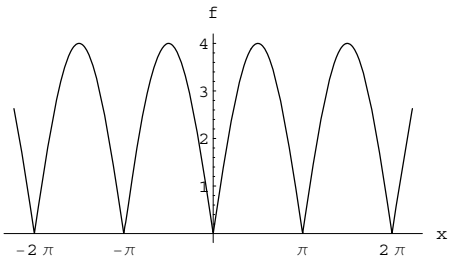
8. The fundamental period is $T = 1$, so $\omega = 2\pi$ and the values of $n\omega$ are $0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots$. From Problem 3, $c_0 = \frac{1}{4}$ and $|c_n| = |(-i)^n - 1|/2n\pi$, or $c_1 = c_{-1} = \sqrt{2}/2\pi$, $c_2 = c_{-2} = 1/2\pi$, $c_3 = c_{-3} = \sqrt{2}/6\pi$, $c_4 = c_{-4} = 0$, $c_5 = c_{-5} = \sqrt{2}/10\pi$, $c_6 = c_{-6} = 1/6\pi$, $c_7 = c_{-7} = \sqrt{2}/14\pi$, $c_8 = c_{-8} = 0, \ldots$. The table shows some values of n with corresponding values of $|c_n|$. The graph is a portion of the frequency spectrum.

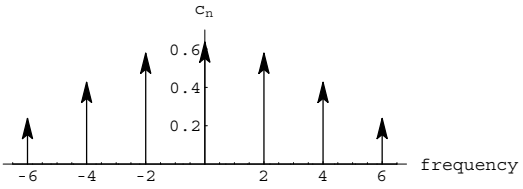


n	-5	-4	-3	-2	-1	0	1	2	3	4	5
c_n	0.0450	0.0000	0.0750	0.1592	0.2251	0.2500	0.2251	0.1592	0.0750	0.0000	0.0450

9. Identifying $2p = \pi$ or $p = \pi/2$, and using $\sin x = (e^{ix} - e^{-ix})/2i$, we have

$$\begin{aligned} c_n &= \frac{1}{\pi}\int_0^{\pi} f(x)e^{-2inx/\pi}dx = \frac{1}{\pi}\int_0^{\pi}(\sin x)e^{-2inx/\pi}dx \\ &= \frac{1}{\pi}\int_0^{\pi}\frac{1}{2i}(e^{ix}-e^{-ix})e^{-2inx/\pi}dx \\ &= \frac{1}{2\pi i}\int_0^{\pi}\left(e^{(1-2n/\pi)ix} - e^{-(1+2n/\pi)ix}\right)dx \\ &= \frac{1}{2\pi i}\left[\frac{1}{i(1-2n/\pi)}e^{(1-2n/\pi)ix} \right. \\ &\qquad \left. + \frac{1}{i(1+2n/\pi)}e^{-(1+2n/\pi)ix}\right]_0^{\pi} \\ &= \frac{\pi(1+e^{-2in})}{\pi^2-4n^2}\,. \end{aligned}$$



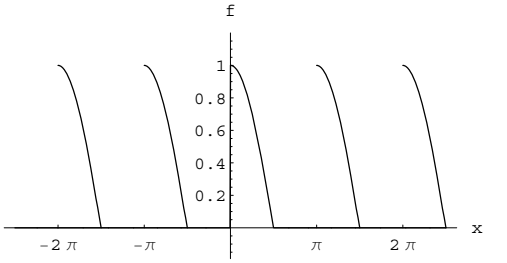


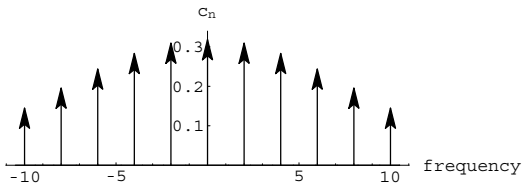
The fundamental period is $T = \pi$, so $\omega = 2\pi/\pi = 2$ and the values of $n\omega$ are 0, ± 2, ± 4, ± 6, ... Values of $|c_n|$ for $n = 0$, ± 1, ± 2, ± 3, ± 4, and ± 5 are shown in the table. The bottom graph is a portion of the frequency spectrum.

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
c_n	0.0198	0.0759	0.2380	0.4265	0.5784	0.6366	0.5784	0.4265	0.2380	0.0759	0.0198

10. Identifying $2p = \pi$ or $p = \pi/2$, and using $\cos x = (e^{ix} - e^{-ix})/2$, we have

$$\begin{aligned} c_n &= \frac{1}{\pi}\int_0^{\pi} f(x)e^{-2inx/\pi}dx \\ &= \frac{1}{\pi}\int_0^{\pi/2}(\cos x)e^{-2inx/\pi}dx \\ &= \frac{1}{\pi}\int_0^{\pi/2}\frac{1}{2}(e^{ix}-e^{-ix})e^{-2inx/\pi}dx \\ &= \frac{1}{2\pi}\int_0^{\pi/2}\left(e^{(1-2n/\pi)ix} - e^{-(1+2n/\pi)ix}\right)dx \\ &= \frac{1}{2\pi}\left[\frac{1}{i(1-2n/\pi)}e^{(1-2n/\pi)ix} \right. \\ &\qquad \left. + \frac{1}{i(1+2n/\pi)}e^{-(1+2n/\pi)ix}\right]_0^{\pi/2} \\ &= \frac{2ne^{-in}+i\pi}{\pi^2-4n^2}\,. \end{aligned}$$





The fundamental period is $T = \pi$, so $\omega = 2\pi/\pi = 2$ and the values of $n\omega$ are 0, ± 2, ± 4, ± 6, ... Values of $|c_n|$ for $n = 0$, ± 1, ± 2, ± 3, ± 4, and ± 5 are shown in the table. The bottom graph is a portion of the frequency spectrum.

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
c_n	0.1447	0.1954	0.2437	0.2833	0.3093	0.3183	0.3093	0.2833	0.2437	0.1954	0.1447

11. **(a)** Adding $c_n = \frac{1}{2}(a_n - ib_n)$ and $c_{-n} = \frac{1}{2}(a_n + ib_n)$ we get $c_n + c_{-n} = a_n$. Subtracting, we get $c_n - c_{-n} = -ib_n$. Multiplying both sides by i we obtain $i(c_n - c_{-n}) = b_n$.

(b) From

$$a_n = c_n + c_{-n} = (-1)^n \frac{\sinh \pi}{\pi}\left[\frac{1-in}{n^2+1} + \frac{1+in}{n^2+1}\right] = \frac{2(-1)^n \sinh \pi}{\pi(n^2+1)}, \quad n = 0, 1, 2, \ldots$$

and

$$b_n = i(c_n - c_{-n}) = i(-1)^n \frac{\sinh \pi}{\pi}\left[\frac{1-in}{n^2+1} - \frac{1+in}{n^2+1}\right] = i(-1)^n \frac{\sinh \pi}{\pi}\left[-\frac{2in}{n^2+1}\right] = \frac{2(-1)^n n \sinh \pi}{\pi(n^2+1)},$$

the Fourier series of f is

$$f(x) = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2+1}\cos nx + \frac{n(-1)^n}{n^2+1}\sin nx\right].$$

12. From Problem 11 and the fact that f is odd, $c_n + c_{-n} = a_n = 0$, so $c_{-n} = -c_n$. Then $b_n = i(c_n - c_{-n}) = 2ic_n$. From Problem 1, $b_n = 2i[1 - (-1)^n]/n\pi i = 2[1 - (-1)^n]/n\pi$, and the Fourier sine series of f is

$$f(x) = \sum_{i=1}^{\infty} \frac{2[1-(-1)^n]}{n\pi} \sin \frac{n\pi x}{2}.$$

EXERCISES 12.5

Sturm-Liouville Problem

1. For $\lambda \leq 0$ the only solution of the boundary-value problem is $y = 0$. For $\lambda = \alpha^2 > 0$ we have

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x.$$

Now

$$y'(x) = -c_1 \alpha \sin \alpha x + c_2 \alpha \cos \alpha x$$

and $y'(0) = 0$ implies $c_2 = 0$, so

$$y(1) + y'(1) = c_1(\cos \alpha - \alpha \sin \alpha) = 0 \quad \text{or} \quad \cot \alpha = \alpha.$$

The eigenvalues are $\lambda_n = \alpha_n^2$ where α_1, α_2, α_3, ... are the consecutive positive solutions of $\cot \alpha = \alpha$. The corresponding eigenfunctions are $\cos \alpha_n x$ for $n = 1, 2, 3, \ldots$. Using a CAS we find that the first four eigenvalues are approximately 0.7402, 11.7349, 41.4388, and 90.8082 with corresponding approximate eigenfunctions $\cos 0.8603x$, $\cos 3.4256x$, $\cos 6.4373x$, and $\cos 9.5293x$.

2. For $\lambda < 0$ the only solution of the boundary-value problem is $y = 0$. For $\lambda = 0$ we have $y = c_1 x + c_2$. Now $y' = c_1$ and the boundary conditions both imply $c_1 + c_2 = 0$. Thus, $\lambda = 0$ is an eigenvalue with corresponding eigenfunction $y_0 = x - 1$.

For $\lambda = \alpha^2 > 0$ we have

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x$$