EXERCISES 7.6

Vector Spaces

1. Not a vector space. Axiom (**vi**) is not satisfied.

2. Not a vector space. Axiom (**i**) is not satisfied.

3. Not a vector space. Axiom (**x**) is not satisfied.

4. A vector space

5. A vector space

6. A vector space

7. Not a vector space. Axiom (**ii**) is not satisfied.

8. A vector space

9. A vector space

10. Not a vector space. Axiom (**i**) is not satisfied.

11. A subspace

12. Not a subspace. Axiom (**i**) is not satisfied.

13. Not a subspace. Axiom (**ii**) is not satisfied.

14. A subspace

15. A subspace

16. A subspace

17. A subspace

18. A subspace

19. Not a subspace. Neither axioms (**i**) nor (**ii**) are satisfied.

20. A subspace

21. Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be in S. Then

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (at_1, bt_1, ct_1) + (at_2, bt_2, ct_2) = (a(t_1 + t_2), b(t_1 + t_2), c(t_1 + t_2))$$

is in S. Also, for (x, y, z) in S then $k(x, y, z) = (kx, ky, kz) = (a(kt), b(kt), c(kt))$ is also in S.

22. Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be in S. Then $ax_1 + by_1 + cz_1 = 0$ and $ax_2 + by_2 + cz_2 = 0$. Adding gives $a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) = 0$ and so $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$ is in S. Also, for (x, y, z) then $ax + by + cz = 0$ implies $k(ax + by + cz) = k \cdot 0 = 0$ and $a(kx) + b(ky) + c(kz) = 0$. this means $k(x, y, z) = (kx, ky, kz)$ is in S.

23. **(a)** $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$ if and only if $c_1 + c_2 + c_3 = 0$, $c_2 + c_3 = 0$, $c_3 = 0$. The only solution of this system is $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

(b) Solving the system $c_1 + c_2 + c_3 = 3$, $c_2 + c_3 = -4$, $c_3 = 8$ gives $c_1 = 7$, $c_2 = -12$, $c_3 = 8$. Thus $\mathbf{a} = 7\mathbf{u}_1 - 12\mathbf{u}_2 + 8\mathbf{u}_3$.

24. **(a)** The assumption $c_1p_1 + c_2p_2 = 0$ is equivalent to $(c_1 + c_2)x + (c_1 - c_2) = 0$. Thus $c_1 + c_2 = 0$, $c_1 - c_2 = 0$. The only solution of this system is $c_1 = 0$, $c_2 = 0$.

(b) Solving the system $c_1 + c_2 = 5$, $c_1 - c_2 = 2$ gives $c_1 = \frac{7}{2}$, $c_2 = \frac{3}{2}$. Thus $p(x) = \frac{7}{2}p_1(x) + \frac{3}{2}p_2(x)$

25. Linearly dependent since $\langle -6, 12\rangle = -\frac{3}{2}\langle 4, -8\rangle$

26. Linearly dependent since $2\langle 1, 1\rangle + 3\langle 0, 1\rangle + (-1)\langle 2, 5\rangle = \langle 0, 0\rangle$

27. Linearly independent

28. Linearly dependent since for all x $(1)\cdot 1+(-2)(x+1)+(1)(x+1)^2+(-1)x^2=0$.

29. f is discontinuous at $x=-1$ and at $x=-3$.

30. $(x,\sin x)=\displaystyle\int_0^{2\pi} x\sin x\,dx=(-x\cos x+\sin x)\,\Big|_0^{2\pi}=-2\pi$

31. $\|x\|^2=\displaystyle\int_0^{2\pi} x^2\,dx=\frac{1}{3}x^3\,\Big|_0^{2\pi}=\frac{8}{3}\pi^3$ and so $\|x\|=2\sqrt{\dfrac{2\pi^3}{3}}$. Now

$$\|\sin x\|^2=\int_0^{2\pi}\sin^2 x\,dx=\frac{1}{2}\int_0^{2\pi}(1-\cos 2x)\,dx=\frac{1}{2}\left(x-\frac{1}{2}\sin 2x\right)\Big|_0^{2\pi}=\pi$$

and so $\|\sin x\|=\sqrt{\pi}$.

32. A basis could be 1, x, $e^x\cos 3x$, $e^x\sin 3x$.

33. We need to show that $\text{Span}\{x_1,x_2,\ldots,x_n\}$ is closed under vector addition and scalar multiplication. Suppose $\mathbf{u}$ and $\mathbf{v}$ are in $\text{Span}\{x_1,x_2,\ldots,x_n\}$. Then $\mathbf{u}=a_1x_1+a_2x_2+\cdots+a_nx_n$ and $\mathbf{v}=b_1x_1+b_2x_2+\cdots+b_nx_n$, so that

$$\mathbf{u}+\mathbf{v}=(a_1+b_1)x_1+(a_2+b_2)x_2+\cdots+(a_n+b_n)x_n,$$

which is in $\text{Span}\{x_1,x_2,\ldots,x_n\}$. Also, for any real number k,

$$k\mathbf{u}=k(a_1x_1+a_2x_2+\cdots+a_nx_n)=ka_1x_1+ka_2x_2+\cdots+ka_nx_n,$$

which is in $\text{Span}\{x_1,x_2,\ldots,x_n\}$. Thus, $\text{Span}\{x_1,x_2,\ldots,x_n\}$ is a subspace of $\mathbf{V}$.

34. R^2 is not a subspace of either R^3 or R^4 and R^3 is not a subspace of R^4. The vectors in R^2 are ordered pairs, while the vectors in R^3 are ordered triples.

35. Since a basis for M_{22} is

$$B=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\},$$

the dimension of M_{22} is 4.

36. To show that the set of nonzero orthogonal vectors is linearly independent we set $c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_n\mathbf{v}_n=\mathbf{0}$. For $0\le i\le n$,

$$(c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_i\mathbf{v}_i\cdots+c_n\mathbf{v}_n)\cdot\mathbf{v}_i=c_1\mathbf{v}_1\cdot\mathbf{v}_i+c_2\mathbf{v}_2\cdot\mathbf{v}_i+\cdots+c_i\mathbf{v}_i\cdot\mathbf{v}_i\cdots+c_n\mathbf{v}_n\cdot\mathbf{v}_i=c_i||\mathbf{v}_i||^2,$$

so $c_i||\mathbf{v}_i||^2=0$ because

$$(c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_i\mathbf{v}_i\cdots+c_n\mathbf{v}_n)\cdot\mathbf{v}_i=\mathbf{0}\cdot\mathbf{v}_i=0.$$

Since $\mathbf{v}_i$ is a nonzero vector, $c_i=0$. Thus, the assumption that $c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_n\mathbf{v}_n=\mathbf{0}$ leads to $c_1=c_2=\cdots=c_n=0$, and the set is linearly independent.

37. We verify the four properties:

(*i*) $(\mathbf{u},\mathbf{v})=u_1v_1+4u_2v_2=v_1u_1+4v_2u_2=(\mathbf{v},\mathbf{u})$

(*ii*) $(k\mathbf{u},\mathbf{v})=(ku_1)v_1+4(ku_2)v_2=k(u_1v_1+4u_2v_2)=k(\mathbf{u},\mathbf{v})$

(*iii*) $(\mathbf{u},\mathbf{u})=u_1^2+4ku_2^2>0$ for $\mathbf{u}\neq\mathbf{0}$. Furthermore, $u_1^2+4ku_2^2=0$ if and only if $u_1=0$ and $u_2=0$, or equivalently, $\mathbf{u}=\mathbf{0}$.

(*iv*) $(\mathbf{u},\mathbf{v}+\mathbf{w})=u_1(v_1+w_1)+4u_2(v_2+w_2)=(u_1v_1+4u_2v_2)+(u_1w_1+4u_2w_2)=(\mathbf{u},\mathbf{v})+(\mathbf{u},\mathbf{w})$

38. (a) Let $\mathbf{u} = \langle 2, 1\rangle$ and $\mathbf{v} = \langle 2, -1\rangle$ be nonzero vectors in R^2. With respect to the standard inner or dot product on R^2,

$$\mathbf{u} \cdot \mathbf{v} = \langle 2, 1\rangle \cdot \langle 2, -1\rangle = 2 \cdot 2 + 1 \cdot (-1) = 3.$$

We see that $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal with respect to that inner product. But using the inner product in Problem 37, we have

$$(\mathbf{u}, \mathbf{v}) = 2 \cdot 2 + 4(1) \cdot (-1) = 0,$$

and so $\mathbf{u}$ and $\mathbf{v}$ are orthogonal with respect to that inner product.

(b) Consider $f(x) = \sin x$ and $g(x) = \cos x$ in $C[0, 2\pi]$. Since

$$\int_0^{2\pi} \sin x \cos x \, dx = \frac{1}{2}\int_0^{2\pi} \sin 2x \, dx = -\frac{1}{4}\cos 2x\Big|_0^{2\pi} = -\frac{1}{4}(1 - 1) = 0,$$

these functions are orthogonal in $C[0, 2\pi]$.

EXERCISES 7.7

Gram-Schmidt Orthogonalization Process

1. Letting $\mathbf{w}_1 = \langle \frac{12}{13}, \frac{5}{13}\rangle$ and $\mathbf{w}_2 = \langle \frac{5}{13}, -\frac{12}{13}\rangle$, we have

$$\mathbf{w}_1 \cdot \mathbf{w}_2 = \left(\frac{12}{13}\right)\left(\frac{5}{13}\right) + \left(\frac{5}{13}\right)\left(-\frac{12}{13}\right) = 0,$$

so the vectors are orthogonal. Also,

$$||\mathbf{w}_1|| = \sqrt{\left(\frac{12}{13}\right)^2 + \left(\frac{5}{13}\right)^2} = 1 \quad \text{and} \quad ||\mathbf{w}_2|| = \sqrt{\left(\frac{5}{13}\right)^2 + \left(-\frac{12}{13}\right)^2} = 1,$$

so the basis is orthonormal. To express $\mathbf{u} = \langle 4, 2\rangle$ in terms of $\mathbf{w}_1$ and $\mathbf{w}_2$ we compute

$$\mathbf{u} \cdot \mathbf{w}_1 = \langle 4, 2\rangle \cdot \left\langle \frac{12}{13}, \frac{5}{13}\right\rangle = (4)\left(\frac{12}{13}\right) + (2)\left(\frac{5}{13}\right) = \frac{58}{13}$$

$$\mathbf{u} \cdot \mathbf{w}_2 = \langle 4, 2\rangle \cdot \left\langle \frac{5}{13}, -\frac{12}{13}\right\rangle = (4)\left(\frac{5}{13}\right) + (2)\left(-\frac{12}{13}\right) = -\frac{4}{13},$$

so

$$\mathbf{u} = \frac{58}{13}\,\mathbf{w}_1 - \frac{4}{13}\,\mathbf{w}_2.$$

2. Letting $\mathbf{w}_1 = \langle 1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}\rangle$, $\mathbf{w}_2 = \langle 0, -1/\sqrt{2}, -1/\sqrt{2}\rangle$, and $\mathbf{w}_3 = \langle -2/\sqrt{6}, 1/\sqrt{6}, -1/\sqrt{6}\rangle$, we have

$$\mathbf{w}_1 \cdot \mathbf{w}_2 = \left(\frac{1}{\sqrt{3}}\right)(0) + \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{2}}\right) = 0$$

$$\mathbf{w}_1 \cdot \mathbf{w}_3 = \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{2}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{6}}\right) + \left(-\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{6}}\right) = 0$$

$$\mathbf{w}_2 \cdot \mathbf{w}_3 = (0)\left(-\frac{2}{\sqrt{6}}\right) + \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{6}}\right) + \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{6}}\right) = 0,$$