17 Functions of a Complex Variable

EXERCISES 17.1

Complex Numbers

1. $3 + 3i$

2. $-4i$

3. $i^8 = (i^2)^4 = (-1)^4 = 1$

4. $i^{11} = i(i^2)^5 = i(-1)^5 = -i$

5. $7 - 13i$

6. $-3 - 9i$

7. $-7 + 5i$

8. $-7 + 8i$

9. $11 - 10i$

10. $\dfrac{3}{4} + \dfrac{2}{3}\,i$

11. $-5 + 12i$

12. $-2 - 2i$

13. $-2i$

14. $\dfrac{i}{1+i} \cdot \dfrac{1-i}{1-i} = \dfrac{i+1}{2} = \dfrac{1}{2} + \dfrac{1}{2}\,i$

15. $\dfrac{2-4i}{3+5i} \cdot \dfrac{3-5i}{3-5i} = \dfrac{-14-22i}{34} = -\dfrac{7}{17} - \dfrac{11}{17}\,i$

16. $\dfrac{10-5i}{6+2i} \cdot \dfrac{6-2i}{6-2i} = \dfrac{50-50i}{40} = \dfrac{5}{4} - \dfrac{5}{4}\,i$

17. $\dfrac{9+7i}{1+i} \cdot \dfrac{1-i}{1-i} = \dfrac{16-2i}{2} = 8 - i$

18. $\dfrac{3-i}{11-2i} \cdot \dfrac{11+2i}{11+2i} = \dfrac{35-5i}{125} = \dfrac{7}{25} - \dfrac{1}{25}\,i$

19. $\dfrac{2-11i}{6-i} \cdot \dfrac{6+i}{6+i} = \dfrac{23-64i}{37} = \dfrac{23}{37} - \dfrac{64}{37}\,i$

20. $\dfrac{4+3i}{3+4i} \cdot \dfrac{3-4i}{3-4i} = \dfrac{24-7i}{25} = \dfrac{24}{25} - \dfrac{7}{25}\,i$

21. $(1+i)(10+10i) = 10(1+i)^2 = 20i$

22. $[(1+i)(1-i)]^2(1-i) = 4 - 4i$

23. $20 + 23i + \dfrac{1}{2-i} \cdot \dfrac{2+i}{2+i} = 20 + 23i + \dfrac{2}{5} + \dfrac{1}{5}\,i = \dfrac{102}{5} + \dfrac{116}{5}\,i$

24. $(2+3i)(-i)^2 = -2 - 3i$

25. $\dfrac{i}{9+7i} \cdot \dfrac{9-7i}{9-7i} = \dfrac{7+9i}{130} = \dfrac{7}{130} + \dfrac{9}{130}\,i$

26. $\dfrac{1}{6+8i} \cdot \dfrac{6-8i}{6-8i} = \dfrac{6-8i}{84} = \dfrac{1}{14} - \dfrac{2}{21}\,i$

27. $\dfrac{x}{x^2+y^2}$

28. x^2-y^2

29. $-2y-4$

30. 0

31. $\sqrt{(x-1)^2+(y-3)^2}$

32. $\sqrt{36x^2+16y^2}$

33. $2x+2yi=-9+2i$ implies $2x=-9$ and $2y=2$. Hence $z=-\frac{9}{2}+i$.

34. $-x+3yi=-7+6i$ implies $-x-7$ and $3y=6$. Hence $z=7+2i$.

35. $x^2-y^2+2xyi=0+i$ implies $x^2-y^2=0$ and $2xy=1$. Now $y=x$ implies $2x^2=1$ and so $x=\pm 1/\sqrt{2}$. The choice $y=-x$ gives $-2x^2=1$ which has no real solution. Hence $z=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\,i$ and $z=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\,i$.

36. $x^2-y^2-4x+(-2xy-4y)i=0+0i$ implies $x^2-y^2-4x=0$ and $y(-2x-4)=0$. If $y=0$ then $x(x-4)=0$ and so $z=0$ and $z=4$. If $-2x-4=0$ or $x=-2$ then $12-y^2=0$ or $y=\pm 2\sqrt{3}$. This gives $z=-2+2\sqrt{3}\,i$ and $z=-2-2\sqrt{3}\,i$.

37. $|10+8i|=\sqrt{164}$ and $|11-6i|=\sqrt{157}$. Hence $11-6i$ is closer to the origin.

38. $|\frac{1}{2}-\frac{1}{4}\,i|=\frac{\sqrt{5}}{4}$ and $|\frac{2}{3}+\frac{1}{6}\,i|=\frac{\sqrt{17}}{6}$. Since $\frac{\sqrt{5}}{4}<\frac{\sqrt{17}}{6}$, $\frac{1}{2}-\frac{1}{4}\,i$ is closer to the origin.

39. $|z_1-z_2|=|(x_1-x_2)+i(y_1-y_2)|=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ which is the distance formula in the plane.

40. By the triangle inequality, $|z+6+8i|\le |z|+|6+8i|$. On the circle, $|z|=2$ and so $|z+6+8i|\le 2+\sqrt{100}=12$.

EXERCISES 17.2

Powers and Roots

1. $2(\cos 2\pi + i\sin 2\pi)$

2. $10(\cos \pi + i\sin \pi)$

3. $3\left(\cos\dfrac{3\pi}{2}+i\sin\dfrac{3\pi}{2}\right)$

4. $6\left(\cos\dfrac{\pi}{2}+i\sin\dfrac{\pi}{2}\right)$

5. $\sqrt{2}\left(\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right)$

6. $5\sqrt{2}\left(\cos\dfrac{7\pi}{4}+i\sin\dfrac{7\pi}{4}\right)$

7. $2\left(\cos\dfrac{5\pi}{6}+i\sin\dfrac{5\pi}{6}\right)$

8. $4\left(\cos\dfrac{4\pi}{3}+i\sin\dfrac{4\pi}{3}\right)$

9. $\dfrac{3\sqrt{2}}{2}\left(\cos\dfrac{5\pi}{4}+i\sin\dfrac{5\pi}{4}\right)$

10. $6\left[\cos\left(-\dfrac{\pi}{6}\right)+i\sin\left(-\dfrac{\pi}{6}\right)\right]$

11. $z=-\dfrac{5\sqrt{3}}{2}-\dfrac{5}{2}\,i$

12. $z=-8+8i$

13. $z=5.5433+2.2961i$

14. $z=8.0902+5.8779i$

15. $z_1z_2=8\left[\cos\left(\dfrac{\pi}{8}+\dfrac{3\pi}{8}\right)+i\sin\left(\dfrac{\pi}{8}+\dfrac{3\pi}{8}\right)\right]=8i;\quad \dfrac{z_1}{z_2}=\dfrac{1}{2}\left[\cos\left(\dfrac{\pi}{8}-\dfrac{3\pi}{8}\right)+i\sin\left(\dfrac{\pi}{8}-\dfrac{3\pi}{8}\right)\right]=\dfrac{\sqrt{2}}{4}-\dfrac{\sqrt{2}}{4}\,i$

16. $z_1 z_2 = \sqrt{6}\left[\cos\left(\frac{\pi}{4}+\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{4}+\frac{\pi}{12}\right)\right] = \frac{\sqrt{6}}{2}+\frac{3\sqrt{2}}{2}\,i$

$\frac{z_1}{z_2} = \frac{\sqrt{6}}{3}\left[\cos\left(\frac{\pi}{4}-\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{4}-\frac{\pi}{12}\right)\right] = \frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{6}\,i$

17. $\left[3\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)\right]\left[10\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right] = 30\sqrt{2}\left[\cos\left(\frac{7\pi}{4}+\frac{\pi}{3}\right)+i\sin\left(\frac{7\pi}{4}+\frac{\pi}{3}\right)\right]$

$= 40.9808 + 10.9808i$

18. $\left[4\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]\left[\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\right] = 8\left[\cos\left(\frac{\pi}{4}+\frac{3\pi}{4}\right)+i\sin\left(\frac{\pi}{4}+\frac{3\pi}{4}\right)\right] = -8$

19. $\dfrac{\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}}{2\sqrt{2}\left[\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right]} = \frac{\sqrt{2}}{4}\left[\cos\left(\frac{3\pi}{2}-\frac{7\pi}{4}\right)+i\sin\left(\frac{3\pi}{2}-\frac{7\pi}{4}\right)\right] = \frac{1}{4}-\frac{1}{4}\,i$

20. $\dfrac{2\sqrt{2}\left[\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right]}{2\left[\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right]} = \sqrt{2}\left[\cos\left(\frac{\pi}{3}-\frac{2\pi}{3}\right)+i\sin\left(\frac{\pi}{3}-\frac{2\pi}{3}\right)\right] = \frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}\,i$

21. $2^9\left[\cos\frac{9\pi}{3}+i\sin\frac{9\pi}{3}\right] = -512$

22. $(2\sqrt{2})^5\left[\cos\left(-\frac{5\pi}{4}\right)+i\sin\left(-\frac{5\pi}{4}\right)\right] = -128+128i$

23. $\left(\frac{\sqrt{2}}{2}\right)^{10}\left[\cos\frac{10\pi}{4}+i\sin\frac{10\pi}{4}\right] = \frac{1}{32}\,i$

24. $(2\sqrt{2})^4\left[\cos\frac{8\pi}{3}+i\sin\frac{8\pi}{3}\right] = -32+32\sqrt{3}\,i$

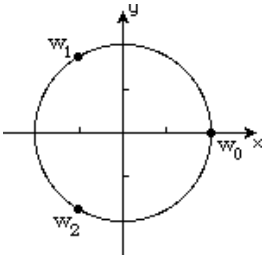
25. $\cos\frac{12\pi}{8}+i\sin\frac{12\pi}{8} = -i$

26. $(\sqrt{3})^6\left[\cos\frac{12\pi}{9}+i\sin\frac{12\pi}{9}\right] = -\frac{27}{2}-\frac{27\sqrt{3}}{2}\,i$

27. $8^{1/3} = 2\left[\cos\frac{2k\pi}{3}+i\sin\frac{2k\pi}{3}\right], \quad k = 0,\ 1,\ 2$

$w_0 = 2[\cos 0 + i\sin 0] = 2; \quad w_1 = 2\left[\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right] = -1+\sqrt{3}\,i$

$w_2 = 2\left[\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right] = -1-\sqrt{3}\,i$



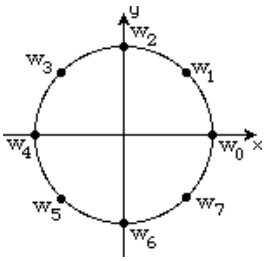
28. $(1)^{1/8} = \cos\frac{k\pi}{4}+i\sin\frac{k\pi}{4}, \quad k = 0,\ 1,\ 2,\ \ldots,\ 7$

$w_0 = \cos 0 + i\sin 0 = 1; \quad w_1 = \cos\frac{\pi}{4}+i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\,i$

$w_2 = \cos\frac{\pi}{2}+i\sin\frac{\pi}{2} = i; \quad w_3 = \cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\,i$

$w_4 = \cos\pi + i\sin\pi = -1; \quad w_5 = \cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\,i$

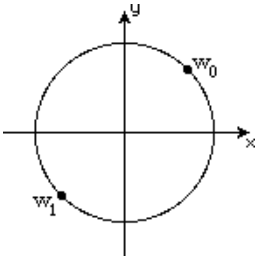
$w_6 = \cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2} = -i; \quad w_7 = \cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4} = \frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\,i$



29. $(i)^{1/2} = \cos\left(\frac{\pi}{4} + k\pi\right) + i\sin\left(\frac{\pi}{4} + k\pi\right), \quad k = 0,\, 1$

$w_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i$

$w_1 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,i$

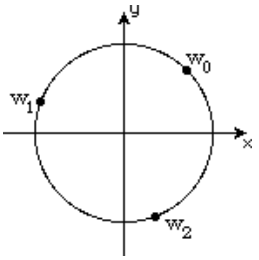


30. $(-1+i)^{1/3} = 2^{1/6}\left[\cos\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right)\right], \quad k = 0,\, 1,\, 2$

$w_0 = 2^{1/6}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{2}}\,i = 0.7937 + 0.7937i$

$w_1 = 2^{1/6}\left[\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right] = -1.0842 + 0.2905i$

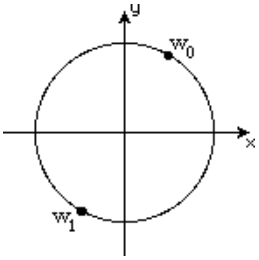
$w_2 = 2^{1/6}\left[\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right] = 0.2905 - 1.0842i$



31. $(-1+\sqrt{3}\,i)^{1/2} = 2^{1/2}\left[\cos\left(\frac{\pi}{3} + k\pi\right) + i\sin\left(\frac{\pi}{3} + k\pi\right)\right], \quad k = 0,\, 1$

$w_0 = 2^{1/2}\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}\,i$

$w_2 = 2^{1/2}\left[\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right] = -\frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}\,i$

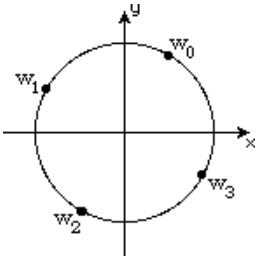


32. $(-1-\sqrt{3}\,i)^{1/4} = 2^{1/4}\left[\cos\left(\frac{\pi}{3} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{3} + \frac{k\pi}{2}\right)\right], \quad k = 0,\, 1,\, 2,\, 3$

$w_0 = 2^{1/4}\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = 2^{1/4}\left[\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right]$

$w_1 = 2^{1/4}\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right] = 2^{1/4}\left[-\frac{\sqrt{3}}{2} + \frac{1}{2}\,i\right]$

$w_2 = 2^{1/4}\left[\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right] = 2^{1/4}\left[-\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right]$; $\quad w_3 = 2^{1/4}\left[\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right] = 2^{1/4}\left[\frac{\sqrt{3}}{2} - \frac{1}{2}\,i\right]$



33. The solutions are the four fourth roots of -1;

$$w_k = \cos\frac{\pi + 2k\pi}{4} + i\sin\frac{\pi + 2k\pi}{4}, \quad k = 0,\, 1,\, 2,\, 3.$$

We have

$w_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i$ $\qquad$ $w_3 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,i$

$w_2 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i$ $\qquad$ $w_4 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,i.$

34. $(z^4-1)^2 = 0$ is the same as $(z-i)^2(z+i)^2(z-1)^2(z+1)^2 = 0$. Thus $z_1 = 1$, $z_2 = -1$, $z_3 = i$, and $z_4 = -i$ are roots of multiplicity two.

35. $\left(\cos\dfrac{\pi}{9}+i\sin\dfrac{\pi}{9}\right)^{12}\left[2\left(\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}\right)\right]^5 = 2^5\left(\cos\dfrac{4\pi}{3}+i\sin\dfrac{4\pi}{3}\right)\left(\cos\dfrac{5\pi}{6}+i\sin\dfrac{5\pi}{6}\right)$

$$= 32\left[\cos\left(\frac{4\pi}{3}+\frac{5\pi}{6}\right)+i\sin\left(\frac{4\pi}{3}+\frac{5\pi}{6}\right)\right]$$

$$= 32\left(\cos\frac{13\pi}{6}+i\sin\frac{13\pi}{6}\right) = 32\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right) = 16\sqrt{3}+16i$$

36. $\dfrac{\left[8\left(\cos\dfrac{3\pi}{8}+i\sin\dfrac{3\pi}{8}\right)\right]^3}{\left[2\left(\cos\dfrac{\pi}{16}+i\sin\dfrac{\pi}{16}\right)\right]^{10}} = \dfrac{2^9}{2^{10}}\left[\cos\left(\dfrac{9\pi}{8}-\dfrac{10\pi}{16}\right)+i\left(\dfrac{9\pi}{8}-\dfrac{10\pi}{16}\right)\right] = \dfrac{1}{2}\left(\cos\dfrac{\pi}{2}+i\sin\dfrac{\pi}{2}\right) = \dfrac{1}{2}\,i$

37. We have

$$(\cos 2\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$

Also

$$(\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + (2\sin\theta\cos\theta)i.$$

Equating real and imaginary parts gives

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

38. We have

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta.$$

Also

$$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3\\ &= \cos^3\theta - 3\cos\theta\sin^2\theta + (3\cos^2\theta\sin\theta - \sin^3\theta)i.\end{aligned}$$

Equating real and imaginary parts gives

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta, \quad \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

39. **(a)** $\operatorname{Arg}(z_1) = \pi,\ \operatorname{Arg}(z_2) = \dfrac{\pi}{2},\ \operatorname{Arg}(z_1z_2) = -\dfrac{\pi}{2},\ \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) = \dfrac{3\pi}{2} \neq \operatorname{Arg}(z_1z_2)$

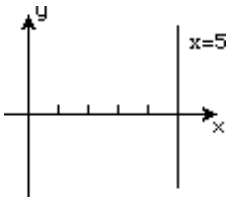
(b) $\operatorname{Arg}(z_1/z_2) = -\dfrac{\pi}{2},\ \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2) = \pi - \dfrac{\pi}{2} = \dfrac{\pi}{2} \neq \operatorname{Arg}(z_1/z_2)$

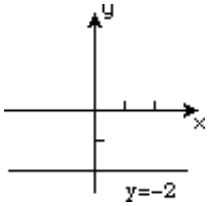
40. **(a)** If we take $\arg(z_1) = \pi$ and $\arg(z_2) = \pi/2$ then $\arg(z_1) + \arg(z_2) = 3\pi/2$ is an argument of the product $z_1z_2 = -5i$. With these same arguments we see that $\arg(z_1) - \arg(z_2) = \pi/2$ is an argument of the quotient $z_1/z_2 = \frac{1}{5}\,i$.

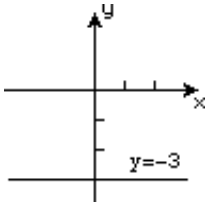
(b) If we take $\arg(z_1) = \pi$ and $\arg(z_2) = -\pi/2$ then $\arg(z_1) + \arg(z_2) = \pi/2$ is an argument of the product $z_1z_2 = 5i$. With these same arguments we see that $\arg(z_1) - \arg(z_2) = 3\pi/2$ is an argument of the quotient $z_1/z_2 = -\frac{1}{5}\,i$.

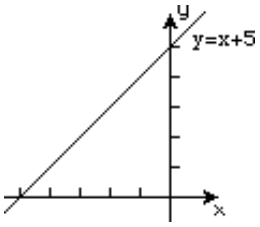
EXERCISES 17.3

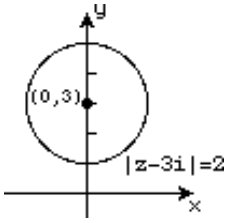
Sets in the Complex Plane

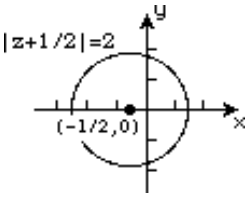
1. 

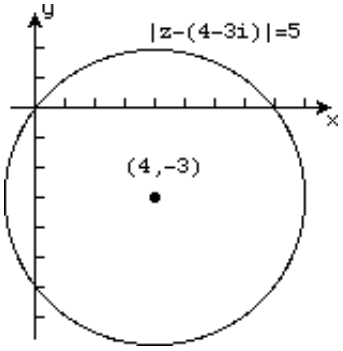
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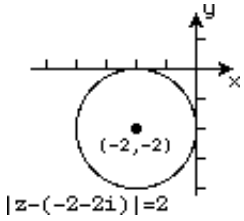
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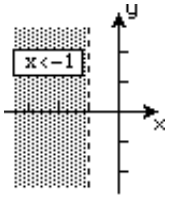
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5. 

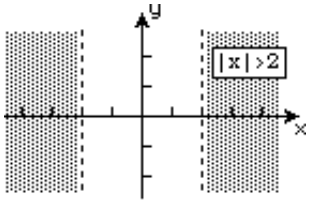
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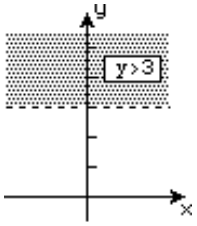
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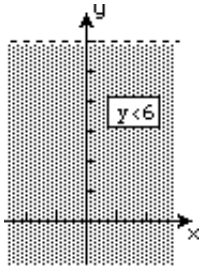
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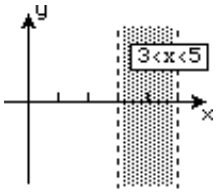
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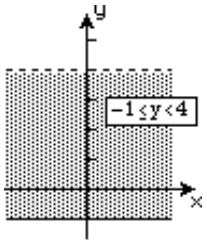
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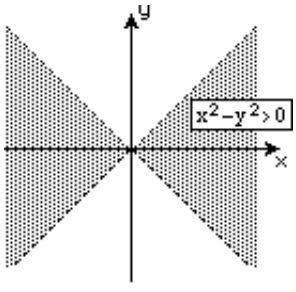
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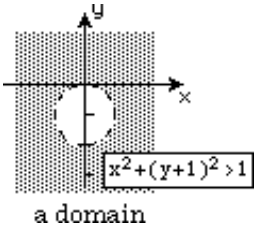
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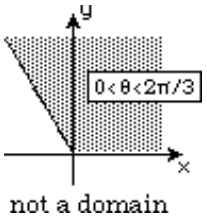
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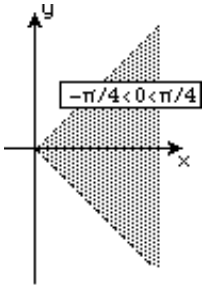
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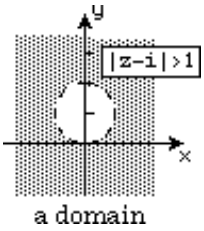
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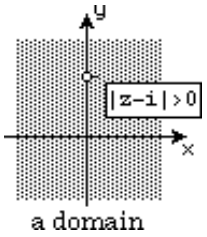
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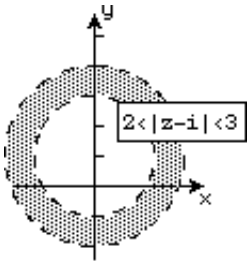
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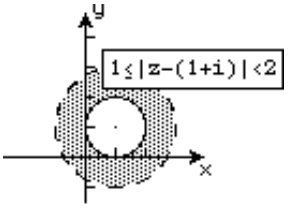
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21.



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22.



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23. The given equation is equivalent to $(x+1)^2 + y^2 = x^2 + (y-1)^2$. This simplifies to $y = -x$ which describes a straight line through the origin.

24. $|\text{Re}(z)| = |x|$ is the same as $\sqrt{x^2}$ and $|z| = \sqrt{x^2+y^2}$. Since $y^2 \geq 0$ the inequality $\sqrt{x^2} \leq \sqrt{x^2+y^2}$ is true for all complex numbers.

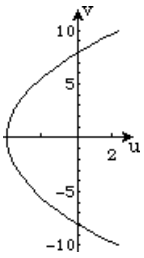
25. The given equation simplifies to the equation $x^2 - y^2 = 1$ which is a hyperbola with center at the origin.

26. Since $|z-i|$ and $|z-(-i)|$ represent distances from the point (x, y) to i and $-i$, respectively, the equation is the distance formula definition of an ellipse with foci at $(0, 1)$ and $(0, -1)$.

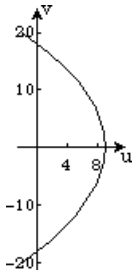
EXERCISES 17.4

Functions of a Complex Variable

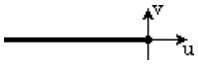
1. Substituting $y = 2$ into $u = x^2 - y^2$, $v = 2xy$ gives the parametric equations $u = x^2 - 4$, $v = 4x$. Using $x = v/4$ the first equation gives $u = v^2/16 - 4$. The graph is the parabola shown.



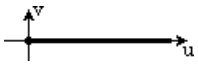
2. Substituting $x = -3$ into $u = x^2 - y^2$, $v = 2xy$ gives the parametric equations $u = 9 - y^2$, $v = -6y$. Using $y = -v/6$ the first equation gives $u = 9 - v^2/36$. The graph is the parabola shown.



3. $x = 0$ gives $u = -y^2$, $v = 0$. Since $-y^2 \le 0$ for all real values of y, the image is the origin and the negative u-axis.



4. $y = 0$ gives $u = x^2$, $v = 0$. Since $x^2 \ge 0$ for all real values of x, the image is the origin and the positive u-axis.



5. $y = x$ gives $u = 0$, $v = 2x^2$. Since $x^2 \ge 0$ for all real values of x, the image is the origin and the positive v-axis.



6. $y = -x$ gives $u = 0$, $v = -2x^2$. Since $-x^2 \le 0$ for all real values of x, the image is the origin and the negative v-axis.



7. $f(z) = (6x - 5) + i(6y + 9)$

8. $f(z) = (7x - 9y - 3) + i(7y - 9x + 2)$

9. $f(z) = (x^2 - y^2 - 3x) + i(2xy - 3y + 4)$

10. $f(z) = (3x^2 - 3y^2 + 2x) + i(-6xy + 2y)$

11. $f(z) = (x^3 - 3xy^2 - 4x) + i(3x^2y - y^3 - 4y)$

12. $f(z) = (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3)$

13. $f(z) = \left(x + \dfrac{x}{x^2+y^2}\right) + i\left(y - \dfrac{y}{x^2+y^2}\right)$ **14.** $f(z) = \dfrac{x^2+y^2+x}{(x+1)^2+y^2} + i\,\dfrac{y}{(x+1)^2+y^2}$

15. **(a)** $f(0+2i) = -4+i$ **(b)** $f(2-i) = 3-9i$ **(c)** $f(5+3i) = 1+86i$

16. **(a)** $f(1+i) = 3-2i$ **(b)** $f(2-i) = \dfrac{7}{2}+10i$ **(c)** $f(1+4i) = 3-32i$

17. **(a)** $f(4-6i) = 14-20i$ **(b)** $f(-5+12i) = -13+43i$ **(c)** $f(2-7i) = 3-26i$

18. **(a)** $f(0+\dfrac{\pi}{4}\,i) = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}\,i$ **(b)** $f(-1-\pi i) = -e^{-1}$ **(c)** $f(3+\dfrac{\pi}{3}\,i) = \dfrac{1}{2}e^3 + \dfrac{\sqrt{3}}{2}\,e^3 i$

19. $\lim\limits_{z\to i}(4z^3 - 5z^2 + 4z + 1 - 5i) = 6 - 5i$

20. $\lim\limits_{z\to 1-i} \dfrac{5z^2-2z+2}{z+1} = \dfrac{5(1-i)^2 - 2(1-i)+2}{2-i} = \dfrac{8}{5} - \dfrac{16}{5}\,i$

21. $\lim\limits_{z\to i} \dfrac{z^4-1}{z-i} = \lim\limits_{z\to i} \dfrac{(z^2-1)(z-i)(z+i)}{z-i} = -4i$

22. $\lim\limits_{z\to 1+i} \dfrac{z^2-2z+2}{z^2-2i} = \lim\limits_{z\to 1+i} \dfrac{[z-(1+i)][z-(1-i)]}{[z-(1+i)][z-(-1-i)]} = \dfrac{1}{2} + \dfrac{1}{2}\,i$

23. Along the y-axis, $\lim\limits_{z\to 0} \dfrac{x+iy}{x-iy} = \lim\limits_{y\to 0} \dfrac{iy}{-iy} = -1$, whereas along the x-axis, $\lim\limits_{z\to 0} \dfrac{x+iy}{x-iy} = \lim\limits_{x\to 0} \dfrac{x}{x} = 1$.

24. Along the line $x = 1$, $\lim\limits_{z\to 1} \dfrac{x+y-1}{z-1} = \lim\limits_{y\to 0} \dfrac{y}{iy} = \dfrac{1}{i} = -i$, whereas along the x-axis, $\lim\limits_{z\to 1} \dfrac{x+y-1}{z-1} = \lim\limits_{x\to 1} \dfrac{x-1}{x-1} = 1$.

25. $f'(z) = \lim\limits_{\Delta z\to 0} \dfrac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim\limits_{\Delta z\to 0} \dfrac{2z\Delta z + (\Delta z)^2}{\Delta z} = \lim\limits_{\Delta z\to 0}(2z + \Delta z) = 2z$

26. $f'(z) = \lim\limits_{\Delta z\to 0} \dfrac{\dfrac{1}{z+\Delta z} - \dfrac{1}{z}}{\Delta z} = \lim\limits_{\Delta z\to 0} \dfrac{-\Delta z}{(\Delta z)z(z+\Delta z)} = \lim\limits_{\Delta z\to 0} \dfrac{-1}{z(z+\Delta z)} = -\dfrac{1}{z^2}$

27. $f'(z) = 12z^2 - (6+2i)z - 5$

28. $f'(z) = 20z^3 - 3iz^2 + (16-2i)z$

29. $f'(z) = (2z+1)(2z-4) + 2(z^2 - 4z + 8i) = 6z^2 - 14z - 4 + 8i$

30. $f'(z) = (z^5 + 3iz^3)(4z^2 + 3iz^2 + 4z - 6i) + (z^4 + iz^3 + 2z^2 - 6iz)(5z^4 + 9iz^2)$

31. $f'(z) = 6z(z^2 - 4i)^2$

32. $f'(z) = 6(2z - 1/z)^5(2 + 1/z^2)$

33. $f'(z) = \dfrac{(2z+i)3 - (3z - 4 + 8i)2}{(2z+1)^2} = \dfrac{8 - 13i}{(2z+i)^2}$

34. $f'(z) = \dfrac{(z^3+1)(10z-1) - (5z^2 - z)3z^2}{(z^3+1)^2} = \dfrac{-5z^4 + 2z^3 + 10z - 1}{(z^3+1)^2}$

35. $3i$ **36.** $0,\ 2-5i$ **37.** $-2i,\ 2i$ **38.** $3-4i,\ 3+4i$

39. We have $\lim\limits_{\Delta z\to 0} \dfrac{\overline{z+\Delta z} - \overline{z}}{\Delta z} = \lim\limits_{\Delta z\to 0} \dfrac{\overline{\Delta z}}{\Delta z}$.

If we let $\Delta z \to 0$ along a horizontal line then $\Delta z = \Delta x$, $\overline{\Delta z} = \Delta x$, and

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1.$$

If we let $\Delta z \to 0$ along a vertical line then $\Delta z = i\Delta y$, $\overline{\Delta z} = -i\Delta y$, and

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1.$$

Since these two limits are not equal, $f(z) = \overline{z}$ cannot be differentiable at any z.

40. We have $$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)(\overline{z} + \overline{\Delta z}) - z\overline{z}}{\Delta z} = \lim_{\Delta z \to 0} \left(\overline{z} + z \frac{\overline{\Delta z}}{\Delta z} + \overline{\Delta z} \right).$$

If $z = 0$, then the above limit becomes

$$f'(0) = \lim_{\Delta z \to 0} \overline{\Delta z} = 0.$$

If $z \neq 0$ then we first let $\Delta z \to 0$ along a horizontal line so that $\Delta z = \Delta x$ and $\overline{\Delta z} = \Delta x$. Thus,

$$f'(z) = \lim_{\Delta z \to 0} \left(\overline{z} + z \frac{\Delta x}{\Delta x} + \Delta x \right) = \overline{z} + z.$$

Next we let $\Delta z \to 0$ along a vertical line so that $\Delta z = i\Delta y$, $\overline{\Delta z} = -i\Delta y$. Thus

$$f'(z) = \lim_{\Delta y \to 0} \left(\overline{z} + z \frac{-i\Delta y}{i\Delta y} + i\Delta y \right) = \overline{z} - z.$$

We must have $\overline{z} + z = \overline{z} - z$ which implies $z = 0$. This is a contradiction to the assumption that $z \neq 0$. Hence $f(z) = |z|^2$ is differentiable only at $z = 0$.

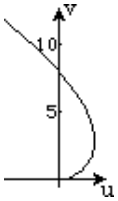
41. Each linear equation in the system $\dfrac{dx}{dt} = 2x$, $\dfrac{dy}{dt} = 2y$ can be solved directly. We obtain $x(t) = c_1 e^{2t}$ and $y(t) = c_2 e^{2t}$.

42. The system $\dfrac{dx}{dt} = -y$, $\dfrac{dy}{dt} = x$ can be solved as in Section 3.11. We obtain $x(t) = c_1 \cos t + c_2 \sin t$, $y(t) = c_1 \sin t - c_2 \cos t$.

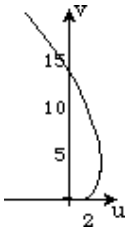
43. The equations in the system $\dfrac{dx}{dt} = \dfrac{x}{x^2 + y^2}$, $\dfrac{dy}{dt} = \dfrac{y}{x^2 + y^2}$ can be divided to give $\dfrac{dy}{dx} = \dfrac{y}{x}$. By separation of variables we obtain $y = cx$.

44. Each equation in the system $\dfrac{dx}{dt} = x^2$, $\dfrac{dy}{dt} = -y^2$ can be solved directly by separation of variables. We obtain $x(t) = \dfrac{-1}{t + c_1}$, $y = \dfrac{1}{t + c_2}$.

45. If $y = \frac{1}{2}x^2$ the equations $u = x^2 - y^2$, $v = 2xy$ give $u = x^2 - \frac{1}{4}x^4$, $v = x^3$. With the aid of a computer, the graph of these parametric equations is shown.



46. If $y = (x - 1)^2$ the equations $u = x^2 - y^2$, $v = 2xy$ give $u = x^2 - (x - 1)^4$, $v = 2x(x - 1)^2$. With the aid of a computer the graph of these parametric equations is shown.



EXERCISES 17.5

Cauchy-Riemann Equations

1. $u = x^3 - 3xy^2,\ v = 3x^2y - y^3;\quad \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2 = \dfrac{\partial v}{\partial y},\ \dfrac{\partial u}{\partial y} = -6xy = -\dfrac{\partial v}{\partial x}$

2. $u = 3x^2 - 3y^2 + 5x,\ v = 6xy + 5y - 6;\quad \dfrac{\partial u}{\partial x} = 6x + 5 = \dfrac{\partial v}{\partial y},\ \dfrac{\partial u}{\partial y} = -6y = -\dfrac{\partial v}{\partial x}$

3. $u = x,\ v = 0;\quad \dfrac{\partial u}{\partial x} = 1,\ \dfrac{\partial v}{\partial y} = 0.$ Since $1 \neq 0$, f is not analytic at any point.

4. $u = y,\quad v = x;\quad \dfrac{\partial u}{\partial x} = 0 = \dfrac{\partial v}{\partial y},\ \dfrac{\partial u}{\partial y} = 1,\ -\dfrac{\partial v}{\partial x} = -1.$ Since $1 \neq -1$, f is not analytic at any point.

5. $u = -2x + 3,\ v = 10y;\quad \dfrac{\partial u}{\partial x} = -2,\ \dfrac{\partial v}{\partial y} = 10.$ Since $-2 \neq 10$, f is not analytic at any point.

6. $u = x^2 - y^2,\ v = -2xy;\quad \dfrac{\partial u}{\partial x} = 2x,\ \dfrac{\partial v}{\partial y} = -2x;\quad \dfrac{\partial u}{\partial y} = -2y,\ -\dfrac{\partial v}{\partial x} = 2y$

The Cauchy-Riemann equations hold only at $(0, 0)$. Since there is no neighborhood about $z = 0$ within which f is differentiable we conclude f is nowhere analytic.

7. $u = x^2 + y^2,\ v = 0;\quad \dfrac{\partial u}{\partial x} = 2x,\ \dfrac{\partial v}{\partial y} = 0;\quad \dfrac{\partial u}{\partial y} = 2y,\ -\dfrac{\partial v}{\partial x} = 0$

The Cauchy-Riemann equations hold only at $(0, 0)$. Since there is no neighborhood about $z = 0$ within which f is differentiable we conclude f is nowhere analytic.

8. $u = \dfrac{x}{x^2 + y^2},\ v = \dfrac{y}{x^2 + y^2};\quad \dfrac{\partial u}{\partial x} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2},\quad \dfrac{\partial v}{\partial y} = \dfrac{x^2 - y^2}{(x^2 + y^2)^2};\quad \dfrac{\partial u}{\partial y} = -\dfrac{2xy}{(x^2 + y^2)^2} = \dfrac{\partial v}{\partial x}$

The Cauchy-Riemann equations hold only at $(0, 0)$. Since there is no neighborhood about $z = 0$ within which f is differentiable, we conclude f is nowhere analytic.

9. $u = e^x \cos y,\ v = e^x \sin y;\quad \dfrac{\partial u}{\partial x} = e^x \cos y = \dfrac{\partial v}{\partial y};\quad \dfrac{\partial u}{\partial y} = -e^x \sin y = -\dfrac{\partial v}{\partial x}.$ f is analytic for all z.

10. $u = x + \sin x \cosh y,\ v = y + \cos x \sinh y;\quad \dfrac{\partial u}{\partial x} = 1 + \cos x \cosh y = \dfrac{\partial v}{\partial y};\quad \dfrac{\partial u}{\partial y} = \sin x \sinh y = -\dfrac{\partial v}{\partial x}.$

f is analytic for all z.

11. $u = e^{x^2 - y^2} \cos 2xy,\ v = e^{x^2 - y^2} \sin 2xy;\quad \dfrac{\partial u}{\partial x} = -2ye^{x^2 - y^2} \sin 2xy + 2xe^{x^2 - y^2} \cos 2xy = \dfrac{\partial v}{\partial y};$

$\dfrac{\partial u}{\partial y} = -2xe^{x^2 - y^2} \sin 2xy - 2ye^{x^2 - y^2} \cos 2xy = -\dfrac{\partial v}{\partial x}.$ f is analytic for all z.

12. $u = 4x^2 + 5x - 4y^2 + 9,\ v = 8xy + 5y - 1;\quad \dfrac{\partial u}{\partial x} = 8x + 5 = \dfrac{\partial v}{\partial y},\ \dfrac{\partial u}{\partial y} = -8y = -\dfrac{\partial v}{\partial x}.$ f is analytic for all z.

13. $u = \dfrac{x - 1}{(x - 1)^2 + y^2},\ v = -\dfrac{y}{(x - 1)^2 + y^2};\quad \dfrac{\partial u}{\partial x} = \dfrac{y^2 - (x - 1)^2}{[(x - 1)^2 + y^2]} = \dfrac{\partial v}{\partial y},\ \dfrac{\partial u}{\partial y} = -\dfrac{2y(x - 1)}{[(x - 1)^2 + y^2]^2} = -\dfrac{\partial v}{\partial x}$

f is analytic in any domain not containing $z = 1$.

14. $u = \dfrac{x^3 + xy^2 + x}{x^2 + y^2},\ v = \dfrac{x^2y + y^3 - y}{x^2 + y^2};\quad \dfrac{\partial u}{\partial x} = \dfrac{x^4 + 2x^2y^2 - x^2 + y^2 + y^4}{(x^2 + y^2)^2} = \dfrac{\partial v}{\partial y},\ \dfrac{\partial u}{\partial y} = \dfrac{-2xy}{(x^2 + y^2)^2} = -\dfrac{\partial v}{\partial x}$

f is analytic in any domain not containing $z = 0$.

15. $\dfrac{\partial u}{\partial x} = 3 = b = \dfrac{\partial v}{\partial y}$; $\dfrac{\partial u}{\partial y} = -1 = -a = -\dfrac{\partial v}{\partial x}$. f is analytic for all z when $b = 3$, $a = 1$.

16. The Cauchy-Riemann equations yield the system

$$\begin{aligned} 2x + ay &= dx + 2y \\ ax + 2by &= -2cx - dy \end{aligned} \quad \text{or} \quad \begin{aligned} (2-d)x + (2-a)y &= 0 \\ (a+2c)x + (2b+d)y &= 0. \end{aligned}$$

The system holds for $z = x + iy$ whenever $2 - d = 0$, $2 - a = 0$, $a + 2c = 0$, and $2b + d = 0$. That is, f is analytic for all z when $a = 2$, $b = -1$, $c = -1$, and $d = 2$.

17. $u = x^2 + y^2$, $v = 2xy$; $\dfrac{\partial u}{\partial x} = 2x$, $\dfrac{\partial v}{\partial y} = 2x$; $\dfrac{\partial u}{\partial y} = 2y$, $-\dfrac{\partial v}{\partial x} = -2y$

u and v are continuous and have continuous first partial derivatives. The Cauchy-Riemann equations are satisfied for any x and for $y = 0$, that is, for points on the real axis. The function f is differentiable but not analytic along this axis; there is no neighborhood about any point $z = x$ within which f is differentiable.

18. $u = 3x^2y^2$, $v = -6x^2y^2$; $\dfrac{\partial u}{\partial x} = 6xy^2$, $\dfrac{\partial v}{\partial y} = -12x^2y$; $\dfrac{\partial u}{\partial y} = 6x^2y$, $-\dfrac{\partial v}{\partial x} = 12xy^2$

u and v are continuous and have continuous first partial derivatives. The Cauchy-Riemann equations are satisfied whenever $6xy(y + 2x) = 0$ and $6xy(x - 2y) = 0$. The point satisfying $y + 2x = 0$ and $x - 2y = 0$ is $z = 0$. The points that satisfy $6xy = 0$ are the points along the y-axis ($x = 0$) or along the x-axis ($y = 0$). The function f is differentiable but not analytic on either axis; there is no neighborhood about any point $z = x$ or $z = iy$ within which f is differentiable.

19. $u = x^3 + 3xy^2 - x$, $v = y^3 + 3x^2y - y$; $\dfrac{\partial u}{\partial x} = 3x^2 + 3y^2 - 1$, $\dfrac{\partial v}{\partial y} = 3y^2 + 3x^2 - 1$; $\dfrac{\partial u}{\partial y} = 6xy$, $-\dfrac{\partial v}{\partial x} = -6xy$.

u and v are continuous and have continuous first partial derivatives. The Cauchy-Riemann equations are satisfied whenever $6xy = -6xy$ or $12xy = 0$. The points satisfying $12xy = 0$ are the points along the y-axis ($x = 0$) or along the x-axis ($y = 0$). The function f is differentiable but not analytic on either axis; there is no neighborhood about any point $z = x$ or $z = iy$ within which f is differentiable.

20. $u = x^2 - x + y$, $v = y^2 - 5y - x$; $\dfrac{\partial u}{\partial x} = 2x - 1$, $\dfrac{\partial v}{\partial y} = 2y - 5$; $\dfrac{\partial u}{\partial y} = 1$, $-\dfrac{\partial v}{\partial x} = 1$

u and v are continuous and have continuous first partial derivatives. The Cauchy-Riemann equations are satisfied whenever $2x - 1 = 2y - 5$ or for points on the line $y = x + 2$. The function f is differentiable but not analytic on this line; there is no neighborhood about any point $z = x + (x + 2)i$ within which f is differentiable.

21. Since f is entire,

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y = f(z).$$

22. Since f is entire,

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = -2ye^{x^2-y^2} \sin 2xy + 2xe^{x^2-y^2} \cos 2xy + i(2ye^{x^2-y^2} \cos 2xy + 2xe^{x^2-y^2} \sin 2xy).$$

23. $\dfrac{\partial^2 u}{\partial x^2} = 0$, $\dfrac{\partial^2 u}{\partial y^2} = 0$ gives $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. Thus u is harmonic. Now $\dfrac{\partial u}{\partial x} = 1 = \dfrac{\partial v}{\partial y}$ implies $v = y + h(x)$, $\dfrac{\partial u}{\partial y} = 0 = -\dfrac{\partial v}{\partial x}$ implies $0 = -h'(x)$, and so $h(x) = C$ (a constant.) Therefore $f(z) = x + i(y + C)$.

24. $\dfrac{\partial^2 u}{\partial x^2} = 0$, $\dfrac{\partial^2 u}{\partial y^2} = 0$ gives $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. Thus u is harmonic. Now $\dfrac{\partial u}{\partial x} = 2 - 2y = \dfrac{\partial v}{\partial y}$ implies $v = 2y - y^2 + h(x)$, $\dfrac{\partial u}{\partial y} = -2x = -\dfrac{\partial v}{\partial x} = -h'(x)$ implies $h'(x) = 2x$ or $h(x) = x^2 + C$. Therefore $f(z) = 2x - 2xy + i(2y - y^2 + x^2 + C)$.

25. $\dfrac{\partial^2 u}{\partial x^2} = 2$, $\dfrac{\partial^2 u}{\partial y^2} = -2$ gives $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. Thus u is harmonic. Now $\dfrac{\partial u}{\partial x} = 2x = \dfrac{\partial v}{\partial y}$ implies $v = 2xy + h(x)$, $\dfrac{\partial u}{\partial y} = -2y = -\dfrac{\partial v}{\partial x} = -2y - h'(x)$ implies $h'(x) = 0$ or $h(x) = C$. Therefore $f(z) = x^2 - y^2 + i(2xy + C)$.

26. $\dfrac{\partial^2 u}{\partial x^2} = -24xy$, $\dfrac{\partial^2 u}{\partial y^2} = 24xy$ gives $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. Thus u is harmonic. Now $\dfrac{\partial u}{\partial x} = 4y^3 - 12x^2y + 1 = \dfrac{\partial v}{\partial y}$ implies $v = y^4 - 6x^2y^2 + y + h(x)$, $\dfrac{\partial u}{\partial y} = 12xy^2 - 4x^3 = -\dfrac{\partial v}{\partial x} = 12xy^2 - h'(x)$ implies $h'(x) = 4x^3$ or $h(x) = x^4 + C$. Therefore $f(z) = 4xy^3 - 4x^3y + x + i(y^4 - 6x^2y^2 + y + x^4 + C)$.

27. $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{2y^2 - 2x^2}{(x^2 + y^2)^2}$, $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{2x^2 - 2y^2}{(x^2 + y^2)^2}$ gives $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. Thus u is harmonic. Now $\dfrac{\partial u}{\partial x} = \dfrac{2x}{x^2 + y^2} = \dfrac{\partial v}{\partial y}$ implies $v = 2\tan^{-1}\dfrac{y}{x} + h(x)$, $\dfrac{\partial u}{\partial y} = \dfrac{2y}{x^2 + y^2} = -\dfrac{\partial v}{\partial x} = \dfrac{2y}{x^2 + y^2} - h'(x)$ implies $h'(x) = 0$ or $h(x) = C$. Therefore $f(z) = \log_e(x^2 + y^2) + i\left(\tan^{-1}\dfrac{y}{x} + C\right)$, $z \neq 0$.

28. $\dfrac{\partial^2 u}{\partial x^2} = 2e^x \cos y + e^x(x\cos y - y\sin y)$, $\dfrac{\partial^2 u}{\partial y^2} = e^x(-x\cos y + y\sin y - 2\cos y)$ gives $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. Thus u is harmonic. Now $\dfrac{\partial u}{\partial x} = e^x \cos y + e^x(x\cos y - y\sin y) = \dfrac{\partial v}{\partial y}$. Integrating by parts with respect to y implies

$$v = e^x \sin y + e^x(x\sin y + y\cos y - \sin y) + h(x) = xe^x \sin y + ye^x \cos y + h(x),$$

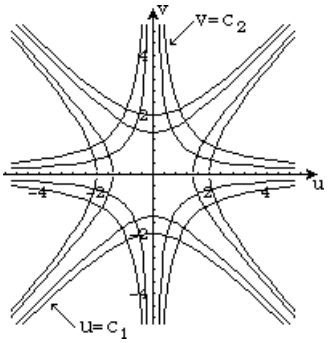
and

$$\frac{\partial u}{\partial y} = -xe^x \sin y - ye^x \cos y - e^x \sin y = -\frac{\partial v}{\partial x} = -xe^x \sin y - e^x \sin y - ye^x \cos y + h'(x)$$

implies $h'(x) = 0$ or $h(x) = C$. Therefore

$$f(z) = e^x(x\cos y - y\sin y) + ie^x(x\sin y + y\cos y + C).$$

29. The level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are the families of hyperbolas $x^2 - y^2 = c_1$ and $2xy = c_2$, respectively. The graphs of these families are displayed on the same axes in the figure.



30. $f(x) = \dfrac{x}{x^2 + y^2} - i\dfrac{y}{x^2 + y^2}$. The level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are the family of circles $x = c_1(x^2 + y^2)$ and $-y = c_2(x^2 + y^2)$, with the exception that $(0, 0)$ is not on the circumference of any circle.

31. $f(z) = x + \dfrac{x}{x^2 + y^2} + i\left(y - \dfrac{y}{x^2 + y^2}\right)$. The level curve $v(x, y) = 0$ is described by $y - \dfrac{y}{x^2 + y^2} = 0$ or $y(x^2 + y^2 - 1) = 0$. We see that either $y = 0$ or $x^2 + y^2 = 1$. Thus $v(x, y) = 0$ gives either the x-axis (without the origin $(0, 0)$) or the unit circle $x^2 + y^2 = 1$.

32. If $\nabla u = \dfrac{\partial u}{\partial x}\,\mathbf{i} + \dfrac{\partial u}{\partial y}\,\mathbf{j}$ and $\nabla v = \dfrac{\partial v}{\partial x}\,\mathbf{i} + \dfrac{\partial v}{\partial y}\,\mathbf{j}$, then $\nabla u \cdot \nabla v = \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial y}$. By the Cauchy-Riemann equations this becomes

$$\nabla u \cdot \nabla v = \frac{\partial v}{\partial y}\frac{\partial v}{\partial x} + \left(-\frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial y} = 0.$$

Since the gradients of u and v are orthogonal vectors, the level curves $u(x, y) = c_1$ and $u(x, y) = c_2$ are orthogonal families.

EXERCISES 17.6

Exponential and Logarithmic Functions

1. $e^{\frac{\pi}{6}i} = \cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}\,i$

2. $e^{-\frac{\pi}{3}i} = \cos\dfrac{\pi}{3} - i\sin\dfrac{\pi}{3} = \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\,i$

3. $e^{-1+\frac{\pi}{4}i} = e^{-1}\cos\dfrac{\pi}{4} + ie^{-1}\sin\dfrac{\pi}{4} = e^{-1}\left(\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}\,i\right)$

4. $e^{2-\frac{\pi}{2}i} = e^2\cos\left(-\dfrac{\pi}{2}\right) + ie^2\sin\left(-\dfrac{\pi}{2}\right) = -e^2 i$

5. $e^{\pi+\pi i} = e^{\pi}\cos\pi + ie^{\pi}\sin\pi = -e^{\pi}$

6. $e^{-\pi+\frac{3\pi}{2}i} = e^{-\pi}\cos\dfrac{3\pi}{2} + ie^{-\pi}\sin\dfrac{3\pi}{2} = -e^{-\pi}i$

7. $e^{1.5+2i} = e^{1.5}\cos 2 + ie^{1.5}\sin 2 = -1.8650 + 4.0752i$

8. $e^{-0.3+0.5i} = e^{-0.3}\cos 0.5 + ie^{-0.3}\sin 0.5 = 0.6501 + 0.3552i$

9. $e^{5i} = \cos 5 + i\sin 5 = 0.2837 - 0.9589i$

10. $e^{-0.23-i} = e^{-0.23}\cos(-1) + ie^{-0.23}\sin(-1) = 0.4293 - 0.6686i$

11. $e^{\frac{11\pi}{12}i} = \cos\dfrac{11\pi}{12} + i\sin\dfrac{11\pi}{12} = -0.9659 + 0.2588i$

12. $e^{5+\frac{5\pi}{2}i} = e^5\cos\dfrac{5\pi}{2} + ie^5\sin\dfrac{5\pi}{2} = e^5 i$

13. $e^{-iz} = e^{y-xi} = e^y\cos x - ie^y\sin x$

14. $e^{2\bar{z}} = e^{2x-2yi} = e^{2x}\cos 2y - ie^{2x}\sin 2y$

15. $e^{z^2} = e^{x^2-y^2+2xyi} = e^{x^2-y^2}\cos 2xy + ie^{x^2-y^2}\sin 2xy$

16. $e^{1/z} = e^{x/(x^2+y^2)-iy/(x^2+y^2)} = e^{x/(x^2+y^2)}\cos\dfrac{y}{x^2+y^2} - ie^{x/(x^2+y^2)}\sin\dfrac{y}{x^2+y^2}$

17. $|e^z|^2 = e^{2x}\cos^2 y + e^{2x}\sin^2 y = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x}$ implies $|e^z| = e^x$.

18. $\dfrac{e^{z_1}}{e^{z_2}} = \dfrac{e^{x_1}\cos y_1 + ie^{x_1}\sin y_1}{e^{x_2}\cos y_2 + ie^{x_2}\sin y_2} = \dfrac{(e^{x_1}\cos y_1 + ie^{x_1}\sin y_1)(e^{x_2}\cos y_2 - ie^{x_2}\sin y_2)}{e^{2x_2}}$

$= e^{x_1-x_2}[(\cos y_1\cos y_2 + \sin y_1\sin y_2) + i(\sin y_1\cos y_2 - \cos y_1\sin y_2)]$

$= e^{x_1-x_2}[\cos(y_1 - y_2) + i\sin(y_1 - y_2)] = e^{x_1-x_2+i(y_1-y_2)} = e^{(x_1+iy_1)-(x_2+iy_2)} = e^{z_1-z_2}$

19. $e^{z+\pi i} = e^{x+(y+\pi)i} = e^x[\cos(y+\pi) + i\sin(y+\pi)] = e^x[\cos(y-\pi) + i\sin(y-\pi)] = e^{x+(y-\pi)i} = e^{z-\pi i}$

20. $(e^z)^n = (e^x[\cos y + i\sin y])^n = e^{nx}[\cos y + i\sin y]^n = e^{nx}[\cos ny + i\sin ny] = e^{nz}$, n an integer

21. $u = e^x\cos y$, $v = -e^x\sin y$; $\dfrac{\partial u}{\partial x} = e^x\cos y$, $\dfrac{\partial v}{\partial y} = -e^x\cos y$; $\dfrac{\partial u}{\partial y} = -e^x\sin y$, $-\dfrac{\partial v}{\partial x} = e^x\sin y$

Since the Cauchy-Riemann equations are not satisfied at any point, f is nowhere analytic.

22. (a) $u = e^{x^2-y^2}\cos 2xy$, $v = e^{x^2-y^2}\sin 2xy$; $\dfrac{\partial u}{\partial x} = -2ye^{x^2-y^2}\sin 2xy + 2xe^{x^2-y^2}\cos 2xy = \dfrac{\partial v}{\partial y}$;

$\dfrac{\partial u}{\partial y} = -2xe^{x^2-y^2}\sin 2xy - 2ye^{x^2-y^2}\cos 2xy = -\dfrac{\partial v}{\partial x}$

Since u, v, and their first partial derivatives are continuous, and u and v satisfy the Cauchy-Riemann equations everywhere, the function f is differentiable everywhere. Hence f is entire.

(b) $\dfrac{\partial^2 u}{\partial x^2} = -4y^2e^{x^2-y^2}\cos 2xy - 4xye^{x^2-y^2}\sin 2xy - 4xye^{x^2-y^2}\sin 2xy + \cos 2xy[4x^2e^{x^2-y^2} + 2e^{x^2-y^2}]$;

$\dfrac{\partial^2 u}{\partial y^2} = -4x^2e^{x^2-y^2}\cos 2xy + 4xye^{x^2-y^2}\sin 2xy + 4xye^{x^2-y^2}\sin 2xy + \cos 2xy[4y^2e^{x^2-y^2} - 2e^{x^2-y^2}]$

Since $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ the function u is harmonic throughout the z-plane.

23. $\ln(-5) = \log_e 5 + i(\pi + 2n\pi) = 1.16094 + (\pi + 2n\pi)i$

24. $\ln(-ei) = \log_e e + i\left(-\dfrac{\pi}{2} + 2n\pi\right) = 1 + \left(-\dfrac{\pi}{2} + 2n\pi\right)i$

25. $\ln(-2+2i) = \log_e 2\sqrt{2} + i\left(\dfrac{3\pi}{4} + 2n\pi\right) = 1.0397 + \left(\dfrac{3\pi}{4} + 2n\pi\right)i$

26. $\ln(1+i) = \log_e\sqrt{2} + i\left(\dfrac{\pi}{4} + 2n\pi\right) = 0.3466 + \left(\dfrac{\pi}{4} + 2n\pi\right)i$

27. $\ln(\sqrt{2} + \sqrt{6}\,i) = \log_e 2\sqrt{2} + i\left(\dfrac{\pi}{3} + 2n\pi\right) = 1.0397 + \left(\dfrac{\pi}{3} + 2n\pi\right)i$

28. $\ln(-\sqrt{3} + i) = \log_e 2 + i\left(\dfrac{5\pi}{6} + 2n\pi\right) = 0.6932 + \left(\dfrac{5\pi}{6} + 2n\pi\right)i$

29. $\operatorname{Ln}(6-6i) = \log_e 6\sqrt{2} + i\left(-\dfrac{\pi}{4}\right) = 2.1383 - \dfrac{\pi}{4}\,i$

30. $\operatorname{Ln}(-e^3) = \log_e e^3 + \pi i = 3 + \pi i$

31. $\operatorname{Ln}(-12+5i) = \log_e 13 + i\left(\tan^{-1}\left(-\dfrac{5}{12}\right) + \pi\right) = 2.5649 + 2.7468i$

32. $\operatorname{Ln}(3-4i) = \log_e 5 + i\tan^{-1}\left(-\dfrac{4}{3}\right) = 1.6094 - 0.9273i$

33. $\operatorname{Ln}(1+\sqrt{3}\,i)^5 = \operatorname{Ln}(16 - 16\sqrt{3}\,i) = \log_e 32 - \dfrac{\pi}{3}\,i = 3.4657 - \dfrac{\pi}{3}\,i$

34. $\operatorname{Ln}(1+i)^4 = \operatorname{Ln}(-4) = \log_e 4 + \pi i = 1.3863 + \pi i$

35. $z = \ln(4i) = \log_e 4 + i\left(\dfrac{\pi}{2} + 2n\pi\right) = 1.3863 + \left(\dfrac{\pi}{2} + 2n\pi\right)i$

36. $\dfrac{1}{z} = \ln(-1) = \log_e 1 + i(\pi + 2n\pi) = (2n+1)\pi i$ and so $z = -\dfrac{i}{(2n+1)\pi}$.

37. $z - 1 = \ln(-ie^2) = \log_e e^2 + i\left(\dfrac{3\pi}{2} + 2n\pi\right) = 2 + \left(\dfrac{3\pi}{2} + 2n\pi\right)i$ and so $z = 3\left(\dfrac{3\pi}{2} + 2n\pi\right)i$.

38. By the quadratic formula, $e^z = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\,i$ or $e^z = -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\,i$. Hence

$$z = \ln\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right) = \left(\frac{2\pi}{3} + 2n\pi\right)i \quad \text{or} \quad z = \ln\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right) = \left(\frac{4\pi}{3} + 2n\pi\right)i.$$

39. $(-i)^{4i} = e^{4i\ln(-i)} = e^{4i[\log_e 1 + i(-\frac{\pi}{2} + 2n\pi)]} = e^{(2-8n)\pi}$

40. $3^{1/\pi} = e^{\frac{i}{\pi}\ln 3} = e^{\frac{i}{\pi}[\log_e 3 + 2n\pi i]} = e^{-2n}\left[\cos\left(\dfrac{1}{\pi}\log_e 3\right) + i\sin\left(\dfrac{1}{\pi}\log_e 3\right)\right] = e^{-2n}[0.9395 + 0.3426i]$

41. $(1+i)^{(1+i)} = e^{(1+i)\ln(1+i)} = e^{(1+i)[\log_e\sqrt{2} + i(\frac{\pi}{4} + 2n\pi)]}$

$$= e^{\log_e\sqrt{2} - (\frac{\pi}{4} + 2n\pi)}\left[\cos\left(\frac{\pi}{4} + \log_e\sqrt{2}\right) + i\sin\left(\frac{\pi}{4} + \log_e\sqrt{2}\right)\right] = e^{-2n\pi}[0.2740 + 0.5837i]$$

42. $(1-i)^{2i} = e^{2i\ln(1-i)} = e^{2i[\log_e\sqrt{2} + i(-\frac{\pi}{4} + 2n\pi)]} = e^{\frac{\pi}{2} - 4n\pi}[\cos(\log_e 2) + i\sin(\log_e 2)] = e^{-4n\pi}[3.7004 + 3.0737i]$

43. $(-1)^{-\frac{2i}{\pi}} = e^{-\frac{2i}{\pi}\mathrm{Ln}(-1)} = e^{-\frac{2i}{\pi}(\pi i)} = e^2 = 7.3891$

44. $(1-i)^{2i} = e^{2i\mathrm{Ln}(1-i)} = e^{2i[\log_e\sqrt{2} - \frac{\pi}{4}i]} = e^{\frac{\pi}{2}}[\cos(\log_e 2) + i\sin(\log_e 2)] = 3.7004 + 3.0737i$

45. If $z_1 = i$ and $z_2 = -1 + i$ then

$$\mathrm{Ln}(z_1 z_2) = \mathrm{Ln}(-1 - i) = \log_e\sqrt{2} - \frac{3\pi}{4}\,i,$$

whereas

$$\mathrm{Ln}z_1 + \mathrm{Ln}z_2 = \frac{\pi}{2}\,i + \left(\log_e\sqrt{2} + \frac{3\pi}{4}\,i\right) = \log_e\sqrt{2} + \frac{5\pi}{4}\,i.$$

46. If $z_1 = -i$ and $z_2 = i$ then

$$\mathrm{Ln}(z_1/z_2) = \mathrm{Ln}(-1) = \pi i, \quad \text{whereas} \quad \mathrm{Ln}z_1 - \mathrm{Ln}z_2 = -\frac{\pi}{2}\,i - \frac{\pi}{2}\,i = -\pi i.$$

47. **(a)** The statement is false.

$$\mathrm{Ln}(-1+i)^2 = \mathrm{Ln}(-2i) = \log_e 2 - \frac{\pi}{2}\,i, \quad \text{whereas} \quad 2\mathrm{Ln}(-1+i) = 2\left(\log_e\sqrt{2} + \frac{3\pi}{4}\,i\right) = \log_e 2 + \frac{3\pi}{2}\,i.$$

(b) The statement is false.

$$\mathrm{Ln}i^3 = \mathrm{Ln}(-i) = -\frac{\pi}{2}\,i, \quad \text{whereas} \quad 3\mathrm{Ln}i = \frac{3\pi}{2}\,i.$$

(c) The statement is true. If we take $\arg(-i) = \dfrac{3\pi}{2}$ then $\ln i^3 = \ln(-i) = \dfrac{3\pi}{2}\,i$ for $n = 0$. Also, $3\ln i = 3\left(\dfrac{\pi}{2}\,i\right)$.

48. **(a)** $(i^i)^2 = (e^{i\ln i})^2 = [e^{-(\frac{\pi}{2} + 2n\pi)}]^2 = e^{-(\pi + 4n\pi)}$ and $i^{2i} = e^{2i\ln i} = e^{-(\pi + 4n\pi)}$

(b) $(i^2)^i = (-1)^i = e^{i\ln(-1)} = e^{-(\pi + 2n\pi)}$, whereas $i^{2i} = e^{-(\pi + 4n\pi)}$

49. Since $|z| = \sqrt{x^2 + y^2}$ and $\mathrm{Arg}z = \tan^{-1}\dfrac{y}{x}$ for $x > 0$ we have

$$\mathrm{Ln}z = \log_e|z| + i\mathrm{Arg}z = \log_e(x^2+y^2)^{1/2} + i\tan^{-1}\frac{y}{x} = \frac{1}{2}\log_e(x^2+y^2) + i\tan^{-1}\frac{y}{x}.$$

50. **(a)** $u = \log_e(x^2+y^2)$; $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{2(y^2-x^2)}{(x^2+y^2)^2}$, $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{2(x^2-y^2)}{(x^2+y^2)^2}$

Since $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ the function u is harmonic in any domain not containing the point $(0,0)$.

(b) $v = \tan^{-1}\dfrac{y}{x}$; $\dfrac{\partial^2 v}{\partial x^2} = \dfrac{2xy}{(x^2+y^2)^2}$, $\dfrac{\partial^2 v}{\partial y^2} = \dfrac{-2xy}{(x^2+y^2)}$

Since $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$ the function v is harmonic in any domain not containing the point $(0,0)$.

EXERCISES 17.7

Trigonometric and Hyperbolic Functions

1. $\cos(3i) = \cosh 3 = 10.0677$

2. $\sin(-2i) = i\sinh(-2) = -3.6269i$

3. $\sin\left(\dfrac{\pi}{4} + i\right) = \sin\dfrac{\pi}{4}\cosh(1) + i\cos\dfrac{\pi}{4}\sinh(1) = 1.0911 + 0.8310i$

4. $\cos(2 - 4i) = \cos(2)\cosh(-4) - \sin(2)\sinh(-4) = -11.3642 - 24.8147i$

5. $\tan(i) = \dfrac{\sin(i)}{\cos(i)} = \dfrac{i\sinh(1)}{\cosh(1)} = 0.7616i$

6. $\cot\left(\dfrac{\pi}{2} + 3i\right) = \dfrac{\cos(\frac{\pi}{2} + 3i)}{\sin(\frac{\pi}{2} + 3i)} = \dfrac{-i\sinh(3)}{\cosh(3)} = -0.9951i$

7. $\sec(\pi + i) = \dfrac{1}{\cos(\pi + i)} = \dfrac{1}{-\cosh(1)} = -0.6481$

8. $\csc(1 + i) = \dfrac{1}{\sin(1 + i)} = \dfrac{1}{\sin(1)\cosh(1) + i\cos(1)\sinh(1)} = 0.6215 - 0.3039i$

9. $\cosh(\pi i) = \cos(i(\pi i)) = \cos(-\pi) = \cos\pi = -1$

10. $\sinh\left(\dfrac{3\pi}{2}i\right) = -i\sin\left[i\left(\dfrac{3\pi}{2}\,i\right)\right] = -i\sin\left(-\dfrac{3\pi}{2}\right) = i\sin\dfrac{3\pi}{2} = -i$

11. $\sinh\left(1 + \dfrac{\pi}{3}\,i\right) = \sinh(1)\cos\dfrac{\pi}{3} + i\cosh(1)\sin\dfrac{\pi}{3} = 0.5876 + 1.3363i$

12. $\cosh(2 + 3i) = \cosh(2)\cos(3) + i\sinh(2)\sin(3) = -3.7245 + 0.5118i$

13. $\sin\left(\dfrac{\pi}{2} + i\ln 2\right) = \sin\dfrac{\pi}{2}\cosh(\ln 2) + i\cos\dfrac{\pi}{2}\sinh(\ln 2) = \dfrac{e^{\ln 2} + e^{\ln 2^{-1}}}{2} = \dfrac{2 + \frac{1}{2}}{2} = \dfrac{5}{4}$

14. $\cos\left(\dfrac{\pi}{2} + i\ln 2\right) = \cos\dfrac{\pi}{2}\cosh(\ln 2) - i\sin\dfrac{\pi}{2}\sinh(\ln 2) = -i\cdot\dfrac{e^{\ln 2} - e^{\ln 2^{-1}}}{2} = -i\cdot\dfrac{2 - \frac{1}{2}}{2} = -\dfrac{3}{4}\,i$

15. $\dfrac{e^{iz} - e^{-iz}}{2i} = 2$ gives $e^{2(iz)} - 4ie^{iz} - 1 = 0$. By the quadratic formula, $e^{iz} = 2i \pm \sqrt{3}\,i$ and so

$$iz = \ln[(2 \pm \sqrt{3}\,)i]$$

$$z = -i\left[\log_e(2 \pm \sqrt{3}\,) + \left(\frac{\pi}{2} + 2n\pi\right)i\right] = \frac{\pi}{2} + 2n\pi - i\log_e(2 \pm \sqrt{3}\,), \quad n = 0,\ \pm 1.,\ \pm 2,\ \ldots\,.$$

16. $\dfrac{e^{iz} + e^{-iz}}{2} = -3i$ gives $e^{2(iz)} + 6ie^{iz} + 1 = 0$. By the quadratic formula, $e^{iz} = -3i \pm \sqrt{10}\,i$ and so $iz = \ln[-3 \pm \sqrt{10}\,)i]$. Hence

$$z = -i\left[\log_e(\sqrt{10} - 3) + \left(\frac{\pi}{2} + 2n\pi\right)i\right] \qquad \text{or} \qquad z = -i\left[\log_e(\sqrt{10} + 3) + \left(\frac{3\pi}{2} + 2n\pi\right)i\right]$$

$$z = \frac{\pi}{2} + 2n\pi - i\log_e(\sqrt{10} - 3) \qquad \text{or} \qquad z = \frac{3\pi}{2} + 2n\pi - i\log_e(\sqrt{10} + 3)$$

$n = 0, \pm 1, \pm 2, \ldots$.

17. $\dfrac{e^z - e^{-z}}{2} = i$ gives $e^{2z} - 2ie^z - 1 = 0$. By the quadratic formula, $e^z = -i$ and so

$$z = \ln(-i) = \log_e 1 + \left(-\frac{\pi}{2} + 2n\pi\right) i = \left(-\frac{\pi}{2} + 2n\pi\right) i, \quad n = 0, \pm 1, \pm 2, \ldots .$$

18. $\dfrac{e^z - e^{-z}}{2} = -1$ gives $e^{2z} + 2e^z - 1 = 0$. By the quadratic formula, $e^z = -1 \pm \sqrt{2}$, and so

$$z = \ln(-1 \pm \sqrt{2}\,)$$
$$z = \log_e(\sqrt{2} - 1) + 2n\pi i \quad \text{or} \quad z = \log_e(\sqrt{2} + 1) + (\pi + 2n\pi)i,$$

$n = 0, \pm 1, \pm 2, \ldots$.

19. $\cos z = \sin z$ gives $\tan z = 1$. One solution is $z = \dfrac{\pi}{4}$. Since $\tan z$ is π-periodic, $z = \dfrac{\pi}{4} + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ are also solutions. That these are the only solutions can be proved by solving

$$\frac{e^{iz} + e^{-iz}}{2} = \frac{e^{iz} - e^{-iz}}{2i}$$

by the method illustrated in Problems 15-18.

20. $\cos z = i \sin z$ gives $e^{iz} + e^{-iz} = e^{iz} - e^{-iz}$ or $e^{-iz} = 0$. Since this last equation has no solutions, the original equation has no solutions.

21. $\cos z = \cosh 2$ implies $\cos x \cosh y - i \sin x \sinh y = \cosh 2 + 0i$ and so we must have $\cos x \cosh y = \cosh 2$ and $\sin x \sinh y = 0$. The last equation has solutions $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, or $y = 0$. For $y = 0$ the first equation becomes $\cos x = \cosh 2$. Since $\cosh 2 > 1$ this equation has no solutions. For $x = n\pi$ the first equation becomes $(-1)^n \cosh y = \cosh 2$. Since $\cosh y > 0$ we see n must be even, say, $n = 2k$, $k = 0, \pm 1, \pm 2, \ldots$. Now $\cosh y = \cosh 2$ implies $y = \pm 2$. Solutions of the original equation are then

$$z = 2k\pi \pm 2i, \quad k = 0, \pm 1, \pm 2, \ldots .$$

22. $\sin z = i \sinh 2$ implies $\sin x \cosh y + i \cos x \sinh y = 0 + i \sinh 2$ and so we must have $\sin x \cosh y = 0$ and $\cos x \sinh y = \sinh 2$. Since $\cosh y > 0$ for all real numbers, the first equation has only the solutions $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. For $x = n\pi$ the second equation becomes $(-1)^n \sinh y = \sinh 2$. If n is even, $\sinh y = \sinh 2$ implies $y = 2$ ($\sinh y$ is one-to-one.) If n is odd, $-\sinh y = \sinh 2$ implies $\sinh y = -\sinh(-2)$ and so $y = -2$. Solutions of the original equation are then

$$z = 2k\pi + 2i, \quad z = (2k+1)\pi - 2i, \quad k = 0, \pm 1, \pm 2, \ldots .$$

23. $\cos z = \dfrac{e^{i(x+iy)} + e^{-i(x+i)}}{2} = \dfrac{1}{2}(e^{-y}e^{ix} + e^y e^{-ix}) = \dfrac{1}{2}[e^{-y}(\cos x + i \sin x) + e^y(\cos x - i \sin x)]$

$$= \cos x \left(\frac{e^y + e^{-y}}{2}\right) - i \sin x \left(\frac{e^y - e^{-y}}{2}\right) = \cos x \cosh y - i \sin x \sinh y$$

24. $\sinh z = \dfrac{e^{x+iy} - e^{-x-iy}}{2} = \dfrac{1}{2}(e^x e^{iy} - e^{-x}e^{-iy}) = \dfrac{1}{2}[e^x(\cos y + i \sin y) - e^{-x}(\cos y - i \sin y)]$

$$= \left(\frac{e^x - e^{-x}}{2}\right) \cos y + i \left(\frac{e^x + e^{-x}}{2}\right) \sin y = \sinh x \cos y + i \cosh x \sin y$$

25. $\cosh z = \dfrac{e^{x+iy} + e^{-x-iy}}{2} = \dfrac{1}{2}(e^x e^{iy} + e^{-x}e^{-iy}) = \dfrac{1}{2}[e^x(\cos y + i \sin y) + e^{-x}(\cos y - i \sin y)]$

$$= \left(\frac{e^x + e^{-x}}{2}\right) \cos y + i \left(\frac{e^x - e^{-x}}{2}\right) \sin y = \cosh x \cos y + i \sinh x \sin y$$

26. $|\sinh z|^2 = \sinh^2 x\cos^2 y + \cosh^2 x\sin^2 y = \sinh^2 x\cos^2 y + (1+\sinh^2 x)\sin^2 y$
$= \sinh^2 x(\cos^2 y + \sin^2 y) + \sin^2 y = \sinh^2 x + \sin^2 y$

27. $|\cosh z|^2 = \cosh^2 x\cos^2 y + \sinh^2 x\sin^2 y = (1+\sinh^2 x)\cos^2 y + \sinh^2 x\sin^2 y$
$= \cos^2 y + \sinh^2 x(\cos^2 y + \sin^2 y) = \cos^2 y + \sinh^2 x$

28. $\cos^2 z + \sin^2 z = \left(\dfrac{e^{1z}+e^{-iz}}{2}\right)^2 + \left(\dfrac{e^{iz}-e^{-iz}}{2i}\right)^2 = \dfrac{1}{4}[e^{2iz}+2+e^{-2iz}-(e^{2iz}-2+e^{-2iz})] = \dfrac{4}{4} = 1$

29. $\cosh^2 z - \sinh^2 z = \left(\dfrac{e^{z}+e^{-z}}{2}\right) - \left(\dfrac{e^{z}-e^{-z}}{2}\right)^2 = \dfrac{1}{4}[e^{2z}+2+e^{-2z}-(e^{2z}-2+e^{-2z})] = \dfrac{4}{4} = 1$

30. $\tan z = \dfrac{\sin z}{\cos z} = \dfrac{\sin z\cos z}{|\cos z|^2} = \dfrac{[\sin x\cosh y + i\cos x\sinh y][\cos x\cosh y + i\sin x\sinh y]}{\cos^2 x + \sinh^2 y}$

$= \dfrac{(\sin x\cos x\cosh^2 y - \sin x\cos x\sinh^2 y)}{\cos^2 x+\sinh^2 y} + i\,\dfrac{\cos^2 x\sinh y\cosh y + \sin^2 x\sinh y\cosh y}{\cos^2 x+\sinh^2 y}$

$= \dfrac{\sin x\cos x(\cosh^2 y - \sinh^2 y)}{\cos^2 x+\sinh^2 y} + i\,\dfrac{\sin y\cosh y(\cos^2 x+\sin^2 x)}{\cos^2 x+\sinh^2 y}$

$= \dfrac{\sin x\cos x}{\cos^2 x+\sinh^2 y} + i\,\dfrac{\sinh y\cosh y}{\cos^2 x+\sinh^2 y} = \dfrac{\sin 2x}{2(\cos^2 x+\sinh^2 y)} + i\,\dfrac{\sinh 2y}{2(\cos^2 x+\sinh^2 y)}$

But

$$2\cos^2 x + 2\sinh^2 y = (2\cos^2 x - 1) + (2\sinh^2 y + 1) = \cos 2x + \cosh 2y.$$

Therefore $\tan z = u + iz$ where

$$u = \frac{\sin 2x}{\cos 2x + \cosh 2y}, \qquad v = \frac{\sinh 2y}{\cos 2x + \cosh 2y}.$$

31. $\tanh(z+\pi i) = \dfrac{\sinh(x+(y+\pi)i)}{\cosh(x+(y+\pi)i)} = \dfrac{\sinh x\cos(y+\pi) + i\cosh x\sin(y+\pi)}{\cosh x\cos(y+\pi) + i\sinh x\sin(y+\pi)}$

$= \dfrac{-[\sinh x\cos y + i\cosh x\sinh y]}{-[\cosh x\cos y + i\sinh x\sin y]} = \dfrac{-\sinh z}{-\cosh z} = \tanh z$

32. **(a)** $\overline{\sin z} = \sin x\cosh y - i\cos x\sinh y = \sin x\cosh(-y) + i\cos x\sinh(-y) = \sin(x-iy) = \sin\bar{z}$

(b) $\overline{\cos z} = \cos x\cosh y + i\sin x\sinh y = \cos x\cosh(-y) - i\sin x\sinh(-y) = \cos(x-iy) = \cos\bar{z}$

EXERCISES 17.8

Inverse Trigonometric and Hyperbolic Functions

1. $\sin^{-1}(-i) = -i\ln(1\pm\sqrt{2}) = \begin{cases} 2n\pi - i\log_e(1+\sqrt{2}) \\ (2n+1)\pi - i\log_e(\sqrt{2}-1) \end{cases}$

Since $\sqrt{2}-1 = 1/(\sqrt{2}+1)$ we can have

$$\sin^{-1}(-i) = \begin{cases} 2n\pi - i\log_e(1+\sqrt{2}) \\ (2n+1)\pi + i\log_e(1+\sqrt{2}). \end{cases}$$

This can be written compactly as

$$\sin^{-1}(-i) = n\pi + (-1)^{n+1} i \log_e(1+\sqrt{2}), \quad k = 0, \pm 1, \pm 2, \ldots .$$

2. $\sin^{-1}\sqrt{2} = -i\ln[\sqrt{2} \pm 1)i] = 2n\pi + \frac{\pi}{2} - i\log_e(\sqrt{2}\pm 1) = 2n\pi + \frac{\pi}{2} \pm i\log_e(1+\sqrt{2}), \quad n = 0, \pm 1, \pm 2, \ldots$

3. $\sin^{-1} 0 = -i\ln(\pm 1) = \begin{cases} 2n\pi + i\log_e 1 \\ (2n+1)\pi + i\log_e 1 \end{cases} = \begin{cases} 2n\pi \\ (2n+1)\pi \end{cases} = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$

4. $\sin^{-1}\dfrac{13}{5} = -i\ln\left[\left(\dfrac{13}{5} \pm \dfrac{12}{5}\right) i\right] = \begin{cases} 2n\pi + \frac{\pi}{2} - i\log_e 5 \\ 2n\pi + \frac{\pi}{2} - i\log_e \frac{1}{5} \end{cases} = 2n\pi + \dfrac{\pi}{2} \pm i\log_e 5, \quad n = 0, \pm 1, \pm 2, \ldots$

5. $\cos^{-1} 2 = -i\ln(2 \pm \sqrt{3}) = \begin{cases} 2n\pi - i\log_e(2+\sqrt{3}) \\ 2n\pi - i\log_e(2-\sqrt{3}) \end{cases}$

Since $2 - \sqrt{3} = 1/(2+\sqrt{3})$ this can be written compactly as

$$\cos^{-1} 2 = 2n\pi \pm i\log_e(2+\sqrt{3}), \quad k = 0, \pm 1, \pm 2, \ldots .$$

6. $\cos^{-1} 2i = -i\ln[(2 \pm \sqrt{5})i] = \begin{cases} 2n\pi - \frac{\pi}{2} + i\log_e(2+\sqrt{5}) \\ 2n\pi + \frac{\pi}{2} - i\log_e(2+\sqrt{5}) \end{cases}, \quad n = 0, \pm 1, \pm 2, \ldots$

7. $\cos^{-1}\dfrac{1}{2} = -i\ln\left(\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}\, i\right) = \begin{cases} 2n\pi + \frac{\pi}{3} - i\log_e 1 \\ 2n\pi + \frac{\pi}{3} - i\log_e 1 \end{cases} = 2n\pi \pm \dfrac{\pi}{2}, \quad n = 0, \pm 1, \pm 2, \ldots$

8. $\cos^{-1}\dfrac{5}{3} = -i\ln\left(\dfrac{5}{3} \pm \dfrac{4}{3}\right) = \begin{cases} 2n\pi - i\log_e 3 \\ 2n\pi - i\log_e \frac{1}{3} \end{cases} = 2n\pi \pm i\log_e 3, \quad n = 0, \pm 1, \pm 2, \ldots$

9. $\tan^{-1} 1 = \dfrac{i}{2}\ln\dfrac{i+1}{i-1} = \dfrac{i}{2}\ln(-i) = -n\pi + \dfrac{\pi}{4} + \dfrac{i}{2}\log_e 1 = \dfrac{\pi}{4} - n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$

Note that this can also be written as $\tan^{-1} 1 = \frac{\pi}{4} + n\pi, \quad n = 0, \pm 1, \pm 2, \ldots .$

10. $\tan^{-1} 3i = \dfrac{i}{2}\ln\left(\dfrac{4i}{-2i}\right) = \dfrac{i}{2}\ln(-2) = -\dfrac{\pi}{2} - n\pi + i\log_e\sqrt{2}, \quad n = 0, \pm 1, \pm 2, \ldots$

11. $\sinh^{-1}\dfrac{4}{3} = \ln\left(\dfrac{4}{3} \pm \dfrac{5}{3}\right) = \begin{cases} \log_e 3 + 2n\pi i \\ \log_e \frac{1}{3} + (2n+1)\pi i \end{cases} = (-1)^n \log_e 3 + n\pi i, \quad n = 0, \pm 1, \pm 2, \ldots$

12. $\cosh^{-1} i = \ln[(1 \pm \sqrt{2})i] = \begin{cases} \log_e(1+\sqrt{2}) + (\frac{\pi}{2} + 2n\pi)i \\ \log_e(\sqrt{2}-1) + (-\frac{\pi}{2} + 2n\pi)i \end{cases}, \quad n = 0, \pm 1, \pm 2, \ldots$

13. $\tanh^{-1}(1+2i) = \dfrac{1}{2}\ln\dfrac{2+2i}{-2i} = \dfrac{1}{2}\ln(-1+i) = \dfrac{1}{2}\left[\log_e\sqrt{2} + \left(\dfrac{3\pi}{4} + 2n\pi\right) i\right] = \dfrac{1}{4}\log_e 2 + \left(\dfrac{3\pi}{8} + n\pi\right) i$

14. $\tanh^{-1}(-\sqrt{3}\, i) = \dfrac{1}{2}\ln\dfrac{1-\sqrt{3}\, i}{1+\sqrt{3}\, i} = \dfrac{1}{2}\ln\left(-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\, i\right) = \dfrac{1}{2}\left[\log_e 1 + \left(\dfrac{4\pi}{3} + 2n\pi\right) i\right] = \left(\dfrac{2\pi}{3} + n\pi\right) i,$

$n = 0, \pm 1, \pm 2, \ldots$

CHAPTER 17 REVIEW EXERCISES

1. 0; 32 **2.** third **3.** $-7/25$ **4.** $-8i$ **5.** $4/5$

6. The closed annular region between the circles $|z+2|=1$ and $|z+2|=3$. These circles have center at $z=-2$.

7. False. $\text{Arg}[(-1+i)+(-1-i)] = \text{Arg}(-2) = \pi$

8. $-5\pi/6$

9. $z = \ln(2i) = \log_e 2 + i\left(\dfrac{\pi}{2} + 2n\pi\right),\ n = 0, \pm 1, \pm 2, \ldots$

10. True

11. $(1+i)^{2+i} = e^{(2+i)[\log_e 2 + \frac{\pi}{4} i]} = e^{(\log_e 2 - \frac{\pi}{4}) + i(\log_e \sqrt{2} + \frac{\pi}{2})} = e^{\log_e 2 - \frac{\pi}{4}} \left[\cos\left(\log_e \sqrt{2} + \dfrac{\pi}{2}\right) + i \sin\left(\log_e \sqrt{2} + \dfrac{\pi}{2}\right)\right]$

$= -0.3097 + 0.8577i$

12. $f(-1+i) = -33 + 26i$

13. False

14. $2\pi i$

15. $\text{Ln}(-ie^3) = \log_e e^3 + \left(-\dfrac{\pi}{2}\right) i = 3 - \dfrac{\pi}{2}\, i$

16. True

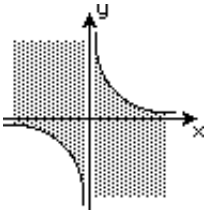
17. $58 - 4i$

18. $-\dfrac{1}{13} - \dfrac{17}{13}\, i$

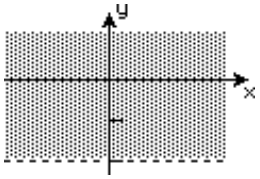
19. $-8 + 8i$

20. $4e^{\pi i/12} = 4\left(\cos\dfrac{\pi}{12} + i\, \sin\dfrac{\pi}{12}\right) = 3.8637 + 1.0353i$

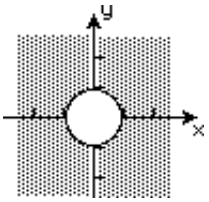
21. The region satisfying $xy \le 1$ is shown in the figure.



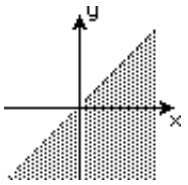
22. The region satisfying $y + 5 > 3$ or $y > -2$ is shown in the figure.



23. The region satisfying $|z| \ge 1$ is shown in the figure.



24. The region satisfying $y < x$ is shown in the figure.



25. Ellipse with foci $(0, -2)$ and $(0, 2)$

26. $\left|\dfrac{z-w}{1-z\bar{w}}\right|^2 = \dfrac{z-w}{1-z\bar{w}} \cdot \dfrac{\bar{z}-\bar{w}}{1-\bar{z}w} = \dfrac{z\bar{z}-z\bar{w}-w\bar{z}+w\bar{w}}{1-\bar{z}w-z\bar{w}+z\bar{z}w\bar{w}} = \dfrac{1-z\bar{w}-w\bar{z}+|w|^2}{1-\bar{z}w-z\bar{w}+|w|^2} = 1,$

since $|z|^2 = z\bar{z} = 1$ and $|w| \neq 1$.

27. The four fourth roots of $1 - i$ are given by

$$w_R = 2^{1/8}\left[\cos\left(-\frac{\pi}{16} + \frac{k\pi}{2}\right) + i\sin\left(-\frac{\pi}{16} + \frac{k\pi}{2}\right)\right], \quad n = 0,\ 1,\ 2,\ 3$$

$$w_0 = 2^{1/8}\left[\cos\left(-\frac{\pi}{16}\right) + i\sin\left(-\frac{\pi}{16}\right)\right] = 1.0696 - 0.2127i$$

$$w_1 = 2^{1/8}\left[\cos\frac{7\pi}{16} + i\sin\frac{7\pi}{16}\right] = 0.2127 + 1.0696i$$

$$w_2 = 2^{1/8}\left[\cos\frac{15\pi}{16} + i\sin\frac{15\pi}{16}\right] = -1.0696 + 0.2127i$$

$$w_3 = 2^{1/8}\left[\cos\frac{23\pi}{16} + i\sin\frac{23\pi}{16}\right] = -0.2127 - 1.0696i$$

28. $z^{3/2} = \frac{2}{5} + \frac{1}{5}\,i$ implies $z^3 = \frac{3}{25} + \frac{4}{25}\,i$. The three cube roots of $\frac{3}{25} + \frac{4}{25}\,i$ are

$$w_k = \left(\frac{1}{5}\right)^{1/3}\left[\cos\left(\frac{1}{3}\tan^{-1}\left(\frac{4}{3}\right) + \frac{2k\pi}{3}\right) + i\sin\left(\frac{1}{3}\tan^{-1}\left(\frac{4}{3}\right) + \frac{2k\pi}{3}\right)\right], \quad k = 0,\ 1,\ 2$$

$$w_0 = \left(\frac{1}{5}\right)^{1/3}[\cos(0.3091) + i\sin(0.3091)] = 0.5571 + 0.1779i$$

$$w_1 = \left(\frac{1}{5}\right)^{1/3}[\cos(2.4035) + i\sin(2.4035)] = -0.4326 + 0.3935i$$

$$w_2 = \left(\frac{1}{5}\right)^{1/3}[\cos(4.4979) + i\sin(4.4979)] = -0.1245 - 0.5714i.$$

29. Write $(1+i)/\sqrt{2} = e^{\pi i/4}$ so that

$$z^{24} = e^{6\pi i} = 1, \quad z^{20} = e^{5\pi i} = -1, \quad z^{12} = e^{3\pi i} = -1, \quad z^6 = e^{3\pi i/2} = -i.$$

Therefore

$$f\left(\frac{1+i}{\sqrt{2}}\right) = 1 - 3(-1) + 4(-1) - 5(-i) = 5i.$$

30. $\text{Im}(z - 3\bar{z}) = 4y$, $z\text{Re}(z^2) = (x^3 - xy^2) + i(x^2y - y^3)$. Thus,

$$f(z) = (4y + x^3 - xy^2 - 5x) + i(x^2y - y^3 - 5y).$$

31. $u = x^2 - y$, $v = y^2 - x$. When $x = 1$ we get the parametric equations $u = 1 - y$, $v = y^2 - 1$. Eliminating y then gives $v = (1-u)^2 - 1 = u^2 - 2u$. This is an equation of a parabola.

32. $u = x/(x^2+y^2)$, $v = -y/(x^2+y^2)$. When $x = 1$ we get the parametric equations $u = 1/(1+y^2)$, $v = -y/(1+y^2)$. From this we find $u^2+v^2-u = 0$. This describes a circle with the exception that $(0, 0)$ is not on its circumference.

33. $z = z^{-1}$ gives $z^2 = 1$ or $(z-1)(z+1) = 0$. Thus $z = \pm 1$.

34. $\bar{z} = 1/z$ gives $z\bar{z} = 1$ or $|z|^2 = 1$. All points on the circle $|z| = 1$ satisfy the equation.

35. $\bar{z} = -z$ gives $x = -x$ or $x = 0$. All complex numbers of the form $z = 0 + iy$ (pure imaginaries) satisfy the equation.

36. $z^2 = \bar{z}^2$ gives $xy = -xy$ or $xy = 0$. This implies $x = 0$ or $y = 0$. All real numbers ($y = 0$) and all pure imaginary numbers ($x = 0$) satisfy the equation.

37. $u = -2xy-5x$, $v = x^2-5y-y^2$; $\dfrac{\partial u}{\partial x} = -2y-5 = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -2x = -\dfrac{\partial v}{\partial x}$; $f'(z) = \dfrac{\partial u}{\partial x} + i\,\dfrac{\partial v}{\partial x} = -2y-5+2xi$

38. $u = x^3 + xy^2 - 4x$, $v = 4y - y^3 - x^2y$; $\dfrac{\partial u}{\partial x} = 3x^2 + y^2 - 4$, $\dfrac{\partial v}{\partial y} = 4 - 3y^2 - x^2$, $\dfrac{\partial u}{\partial y} = 2xy = -\dfrac{\partial v}{\partial x}$

The Cauchy-Riemann equations are satisfied at all points on the circle $x^2 + y^2 = 2$. Continuity of u, v, and the first partial derivatives guarantee f is differentiable on the circle. However, f is nowhere analytic.

39. $\text{Ln}(1+i)(1-i) = \text{Ln}(2) = \log_e 2$; $\text{Ln}(1+i) = \log_e \sqrt{2} + \dfrac{\pi}{4}\,i$; $\text{Ln}(1-i) = \log_e \sqrt{2} - \dfrac{\pi}{4}\,i$.
Therefore,

$$\text{Ln}(1+i) + \text{Ln}(1-i) = 2\log_e \sqrt{2} = \log_e 2 = \text{Ln}(1+i)(1-i).$$

40. $\text{Ln}\dfrac{1+i}{1-i} = \text{Ln}\ i = \log_e 1 + \dfrac{\pi}{2}\,i = \dfrac{\pi}{2}\,i$; $\text{Ln}(1+i) = \log_e \sqrt{2} + \dfrac{\pi}{4}\,i$; $\text{Ln}(1-i) = \log_e \sqrt{2} - \dfrac{\pi}{4}\,i$.
Therefore,

$$\text{Ln}(1+i) - \text{Ln}(1-i) = \frac{\pi}{4}\,i - \left(-\frac{\pi}{4}\,i\right) = \frac{\pi}{2}\,i = \text{Ln}\frac{1+i}{1-i}\,.$$