Exercícios e Soluções

II - Funções de Variáveis Complexas

Obra de referência: Matemática Avançada para Engenharia, Dennis Zill (Vol 3)

Exercícios: Operações Aritméticas

Nos Problemas 1-26, escreva o número indicado na forma $a + ib$.

1. $2i^3 - 3i^2 + 5i$
2. $3i^5 - i^4 + 7i^3 - 10i^2 - 9$
3. i^8
4. i^{11}
5. $(5 - 9i) + (2 - 4i)$
6. $3(4 - i) - 3(5 + 2i)$
7. $i(5 + 7i)$
8. $i(4 - i) + 4i(1 + 2i)$
9. $(2 - 3i)(4 + i)$
10. $(\frac{1}{2} - \frac{1}{4}i)(\frac{2}{3} + \frac{5}{3}i)$
11. $(2 + 3i)^2$
12. $(1 - i)^3$
13. $\dfrac{2}{i}$
14. $\dfrac{i}{1 + i}$
15. $\dfrac{2 - 4i}{3 + 5i}$
16. $\dfrac{10 - 5i}{6 + 2i}$
17. $\dfrac{(3 - i)(2 + 3i)}{1 + i}$
18. $\dfrac{(1 + i)(1 - 2i)}{(2 + i)(4 - 3i)}$
19. $\dfrac{(5 - 4i) - (3 + 7i)}{(4 + 2i) + (2 - 3i)}$
20. $\dfrac{(4 + 5i) + 2i^3}{(2 + i)^2}$
21. $i(1 - i)(2 - i)(2 + 6i)$
22. $(1 + i)^2(1 - i)^3$
23. $(3 + 6i) + (4 - i)(3 + 5i) + \dfrac{1}{2 - i}$
24. $(2 + 3i)\left(\dfrac{2 - i}{1 + 2i}\right)^2$
25. $\left(\dfrac{i}{3 - i}\right)\left(\dfrac{1}{2 + 3i}\right)$
26. $\dfrac{1}{(1 + i)(1 - 2i)(1 + 3i)}$

Nos Problemas 27-32, seja $z = x + iy$. Determine a expressão indicada.

27. $\mathrm{Re}(1/z)$
28. $\mathrm{Re}(z^2)$
29. $\mathrm{Im}(2z + 4\bar{z} - 4i)$
30. $\mathrm{Im}(\bar{z}^2 + z^2)$
31. $|z - 1 - 3i|$
32. $|z + 5\bar{z}|$

Nos Problemas 33-36, use a Definição 8.2 para obter um número complexo z que satisfaça a equação indicada.

33. $2z = i(2 + 9i)$
34. $z - 2\bar{z} + 7 - 6i = 0$
35. $z^2 = i$
36. $\bar{z}^2 = 4z$

Nos Problemas 37 e 38, determine qual número complexo está mais próximo da origem.

37. $10 + 8i,\ 11 - 6i$
38. $\frac{1}{2} - \frac{1}{4}i,\ \frac{2}{3} + \frac{1}{6}i$
39. Prove que $|z_1 - z_2|$ é a distância entre os pontos z_1 e z_2 no plano complexo.
40. Mostre para todos os números complexos z no círculo $x^2 + y^2 = 4$ que $|z + 6 + 8i| \leq 12$.

Soluções

1. $3+3i$

2. $-4i$

3. $i^8 = (i^2)^4 = (-1)^4 = 1$

4. $i^{11} = i(i^2)^5 = i(-1)^5 = -i$

5. $7-13i$

6. $-3-9i$

7. $-7+5i$

8. $-7+8i$

9. $11-10i$

10. $\dfrac{3}{4}+\dfrac{2}{3}\,i$

11. $-5+12i$

12. $-2-2i$

13. $-2i$

14. $\dfrac{i}{1+i}\cdot\dfrac{1-i}{1-i} = \dfrac{i+1}{2} = \dfrac{1}{2}+\dfrac{1}{2}\,i$

15. $\dfrac{2-4i}{3+5i}\cdot\dfrac{3-5i}{3-5i} = \dfrac{-14-22i}{34} = -\dfrac{7}{17}-\dfrac{11}{17}\,i$

16. $\dfrac{10-5i}{6+2i}\cdot\dfrac{6-2i}{6-2i} = \dfrac{50-50i}{40} = \dfrac{5}{4}-\dfrac{5}{4}\,i$

17. $\dfrac{9+7i}{1+i}\cdot\dfrac{1-i}{1-i} = \dfrac{16-2i}{2} = 8-i$

18. $\dfrac{3-i}{11-2i}\cdot\dfrac{11+2i}{11+2i} = \dfrac{35-5i}{125} = \dfrac{7}{25}-\dfrac{1}{25}\,i$

19. $\dfrac{2-11i}{6-i}\cdot\dfrac{6+i}{6+i} = \dfrac{23-64i}{37} = \dfrac{23}{37}-\dfrac{64}{37}\,i$

20. $\dfrac{4+3i}{3+4i}\cdot\dfrac{3-4i}{3-4i} = \dfrac{24-7i}{25} = \dfrac{24}{25}-\dfrac{7}{25}\,i$

21. $(1+i)(10+10i) = 10(1+i)^2 = 20i$

22. $[(1+i)(1-i)]^2(1-i) = 4-4i$

23. $20+23i+\dfrac{1}{2-i}\cdot\dfrac{2+i}{2+i} = 20+23i+\dfrac{2}{5}+\dfrac{1}{5}\,i = \dfrac{102}{5}+\dfrac{116}{5}\,i$

24. $(2+3i)(-i)^2 = -2-3i$

25. $\dfrac{i}{9+7i}\cdot\dfrac{9-7i}{9-7i} = \dfrac{7+9i}{130} = \dfrac{7}{130}+\dfrac{9}{130}\,i$

26. $\dfrac{1}{6+8i}\cdot\dfrac{6-8i}{6-8i} = \dfrac{6-8i}{84} = \dfrac{1}{14}-\dfrac{2}{21}\,i$

Soluções

27. $\dfrac{x}{x^2+y^2}$

28. x^2-y^2

29. $-2y-4$

30. 0

31. $\sqrt{(x-1)^2+(y-3)^2}$

32. $\sqrt{36x^2+16y^2}$

33. $2x+2yi=-9+2i$ implies $2x=-9$ and $2y=2$. Hence $z=-\frac{9}{2}+i$.

34. $-x+3yi=-7+6i$ implies $-x-7$ and $3y=6$. Hence $z=7+2i$.

35. $x^2-y^2+2xyi=0+i$ implies $x^2-y^2=0$ and $2xy=1$. Now $y=x$ implies $2x^2=1$ and so $x=\pm 1/\sqrt{2}$. The choice $y=-x$ gives $-2x^2=1$ which has no real solution. Hence $z=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\,i$ and $z=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\,i$.

36. $x^2-y^2-4x+(-2xy-4y)i=0+0i$ implies $x^2-y^2-4x=0$ and $y(-2x-4)=0$. If $y=0$ then $x(x-4)=0$ and so $z=0$ and $z=4$. If $-2x-4=0$ or $x=-2$ then $12-y^2=0$ or $y=\pm 2\sqrt{3}$. This gives $z=-2+2\sqrt{3}\,i$ and $z=-2-2\sqrt{3}\,i$.

37. $|10+8i|=\sqrt{164}$ and $|11-6i|=\sqrt{157}$. Hence $11-6i$ is closer to the origin.

38. $|\frac{1}{2}-\frac{1}{4}\,i|=\frac{\sqrt{5}}{4}$ and $|\frac{2}{3}+\frac{1}{6}\,i|=\frac{\sqrt{17}}{6}$. Since $\frac{\sqrt{5}}{4}<\frac{\sqrt{17}}{6}$, $\frac{1}{2}-\frac{1}{4}\,i$ is closer to the origin.

39. $|z_1-z_2|=|(x_1-x_2)+i(y_1-y_2)|=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ which is the distance formula in the plane.

40. By the triangle inequality, $|z+6+8i|\le |z|+|6+8i|$. On the circle, $|z|=2$ and so $|z+6+8i|\le 2+\sqrt{100}=12$.

Exercícios: Forma Polar

Nos Problemas 1-10, escreva o número complexo indicado na forma polar.

1. 2
2. -10
3. $-3i$
4. $6i$
5. $1 + i$
6. $5 - 5i$
7. $-\sqrt{3} + i$
8. $-2 - 2\sqrt{3}i$
9. $\dfrac{3}{-1 + i}$
10. $\dfrac{12}{\sqrt{3} + i}$

Nos Problemas 11-14, escreva o número indicado em notação polar na forma $a + ib$.

11. $z = 5\left(\cos\dfrac{7\pi}{6} + i\,\text{sen}\,\dfrac{7\pi}{6}\right)$

12. $z = 8\sqrt{2}\left(\cos\dfrac{11\pi}{4} + i\,\text{sen}\,\dfrac{11\pi}{4}\right)$

13. $z = 6\left(\cos\dfrac{\pi}{8} + i\,\text{sen}\,\dfrac{\pi}{8}\right)$

14. $z = 10\left(\cos\dfrac{\pi}{5} + i\,\text{sen}\,\dfrac{\pi}{5}\right)$

Exercícios: Multiplicação, Divisão, Potenciação e Radiciação usando a Forma Polar

Nos Problemas 15 e 16, determine $z_1 z_2$ e z_1/z_2. Escreva o número na forma $a + ib$.

15. $z_1 = 2\left(\cos\frac{\pi}{8} + i\,\text{sen}\,\frac{\pi}{8}\right),\ z_2 = 4\left(\cos\frac{3\pi}{8} + i\,\text{sen}\,\frac{3\pi}{8}\right)$

16. $z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\,\text{sen}\,\frac{\pi}{4}\right),$

$z_2 = \sqrt{3}\left(\cos\frac{\pi}{12} + i\,\text{sen}\,\frac{\pi}{12}\right)$

Nos Problemas 17-20, escreva cada número complexo na forma polar. A seguir, use (4) ou (5) para obter a forma polar do número indicado. Escreva a forma polar na forma $a + ib$.

17. $(3 - 3i)(5 + 5\sqrt{3}i)$

18. $(4 + 4i)(-1 + i)$

19. $\dfrac{-i}{2 - 2i}$

20. $\dfrac{\sqrt{2} + \sqrt{6}i}{-1 + \sqrt{3}i}$

Exercícios: Multiplicação, Divisão, Potenciação e Radiciação usando a Forma Polar

Nos Problemas 21-26, use (8) para calcular a potência indicada.

21. $(1 + \sqrt{3}i)^9$

22. $(2 - 2i)^5$

23. $(\frac{1}{2} + \frac{1}{2}i)^{10}$

24. $(-\sqrt{2} + \sqrt{6}i)^4$

25. $\left(\cos\frac{\pi}{8} + i\,\text{sen}\,\frac{\pi}{8}\right)^{12}$

26. $\left[\sqrt{3}\left(\cos\frac{2\pi}{9} + i\,\text{sen}\,\frac{2\pi}{9}\right)\right]^6$

Nos Problemas 27-32, aplique (10) para calcular todas as raízes. Esboce essas raízes em um círculo apropriado centrado na origem.

27. $(8)^{1/3}$

28. $(1)^{1/8}$

29. $(i)^{1/2}$

30. $(-1 + i)^{1/3}$

31. $(-1 + \sqrt{3}i)^{1/2}$

32. $(-1 - \sqrt{3}i)^{1/4}$

Nos Problemas 33 e 34, calcule todas as soluções da equação indicada.

33. $z^4 + 1 = 0$

34. $z^8 - 2z^4 + 1 = 0$

Exercícios: Multiplicação, Divisão, Potenciação e Radiciação usando a Forma Polar

Nos Problemas 35 e 36, escreva o número complexo indicado primeiro na forma polar e a seguir na forma $a + ib$.

35. $\left(\cos\frac{\pi}{9} + i\,\text{sen}\,\frac{\pi}{9}\right)^{12}\left[2\left(\cos\frac{\pi}{6} + i\,\text{sen}\,\frac{\pi}{6}\right)\right]^{5}$

36. $\dfrac{\left[8\left(\cos\frac{3\pi}{8} + i\,\text{sen}\,\frac{3\pi}{8}\right)\right]^{3}}{\left[2\left(\cos\frac{\pi}{16} + i\,\text{sen}\,\frac{\pi}{16}\right)\right]^{10}}$

37. Utilize o resultado $(\cos\theta + i\,\text{sen}\,\theta)^2 = \cos 2\theta + i\,\text{sen}\,2\theta$ para obter identidades trigonométricas para $\cos 2\theta$ e $\text{sen}\,2\theta$.

38. Utilize o resultado $(\cos\theta + i\,\text{sen}\,\theta)^3 = \cos 3\theta + i\,\text{sen}\,3\theta$ para obter identidades trigonométricas para $\cos 3\theta$ e $\text{sen}\,3\theta$.

39. (a) Se $z_1 = -1$ e $z_2 = 5i$, verifique que $\text{Arg}(z_1z_2) \neq \text{Arg}(z_1) + \text{Arg}(z_2)$

    (b) Se $z_1 = -1$ e $z_2 = -5i$, verifique que $\text{Arg}(z_1/z_2) \neq \text{Arg}(z_1) - \text{Arg}(z_2)$

40. Para os números complexos indicados no Problema 39, verifique nos itens (a) e (b) que

$$\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$$

e

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2).$$

Soluções

1. $2(\cos 2\pi + i \sin 2\pi)$

2. $10(\cos \pi + i \sin \pi)$

3. $3\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right)$

4. $6\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$

5. $\sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$

6. $5\sqrt{2}\left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}\right)$

7. $2\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$

8. $4\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$

9. $\frac{3\sqrt{2}}{2}\left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right)$

10. $6\left[\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right]$

11. $z = -\frac{5\sqrt{3}}{2} - \frac{5}{2}i$

12. $z = -8 + 8i$

13. $z = 5.5433 + 2.2961i$

14. $z = 8.0902 + 5.8779i$

15. $z_1 z_2 = 8\left[\cos\left(\frac{\pi}{8} + \frac{3\pi}{8}\right) + i \sin\left(\frac{\pi}{8} + \frac{3\pi}{8}\right)\right] = 8i; \quad \frac{z_1}{z_2} = \frac{1}{2}\left[\cos\left(\frac{\pi}{8} - \frac{3\pi}{8}\right) + i \sin\left(\frac{\pi}{8} - \frac{3\pi}{8}\right)\right] = \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4}i$

Soluções

16. $z_1 z_2 = \sqrt{6}\left[\cos\left(\frac{\pi}{4}+\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{4}+\frac{\pi}{12}\right)\right] = \frac{\sqrt{6}}{2}+\frac{3\sqrt{2}}{2}\,i$

$\frac{z_1}{z_2} = \frac{\sqrt{6}}{3}\left[\cos\left(\frac{\pi}{4}-\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{4}-\frac{\pi}{12}\right)\right] = \frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{6}\,i$

17. $\left[3\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)\right]\left[10\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right] = 30\sqrt{2}\left[\cos\left(\frac{7\pi}{4}+\frac{\pi}{3}\right)+i\sin\left(\frac{7\pi}{4}+\frac{\pi}{3}\right)\right]$

$= 40.9808 + 10.9808i$

18. $\left[4\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]\left[\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\right] = 8\left[\cos\left(\frac{\pi}{4}+\frac{3\pi}{4}\right)+i\sin\left(\frac{\pi}{4}+\frac{3\pi}{4}\right)\right] = -8$

19. $\dfrac{\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}}{2\sqrt{2}\left[\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right]} = \frac{\sqrt{2}}{4}\left[\cos\left(\frac{3\pi}{2}-\frac{7\pi}{4}\right)+i\sin\left(\frac{3\pi}{2}-\frac{7\pi}{4}\right)\right] = \frac{1}{4}-\frac{1}{4}\,i$

20. $\dfrac{2\sqrt{2}\left[\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right]}{2\left[\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right]} = \sqrt{2}\left[\cos\left(\frac{\pi}{3}-\frac{2\pi}{3}\right)+i\sin\left(\frac{\pi}{3}-\frac{2\pi}{3}\right)\right] = \frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}\,i$

21. $2^9\left[\cos\frac{9\pi}{3}+i\sin\frac{9\pi}{3}\right] = -512$

22. $(2\sqrt{2})^5\left[\cos\left(-\frac{5\pi}{4}\right)+i\sin\left(-\frac{5\pi}{4}\right)\right] = -128+128i$

23. $\left(\frac{\sqrt{2}}{2}\right)^{10}\left[\cos\frac{10\pi}{4}+i\sin\frac{10\pi}{4}\right] = \frac{1}{32}\,i$

24. $(2\sqrt{2})^4\left[\cos\frac{8\pi}{3}+i\sin\frac{8\pi}{3}\right] = -32+32\sqrt{3}\,i$

25. $\cos\frac{12\pi}{8}+i\sin\frac{12\pi}{8} = -i$

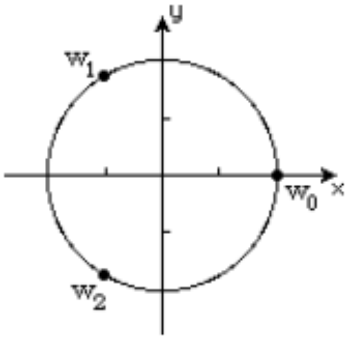
26. $(\sqrt{3})^6\left[\cos\frac{12\pi}{9}+i\sin\frac{12\pi}{9}\right] = -\frac{27}{2}-\frac{27\sqrt{3}}{2}\,i$

Soluções

27. $8^{1/3} = 2\left[\cos\dfrac{2k\pi}{3} + i\sin\dfrac{2k\pi}{3}\right], \quad k = 0,\ 1,\ 2$

$$w_0 = 2[\cos 0 + i\sin 0] = 2; \quad w_1 = 2\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right] = -1 + \sqrt{3}\,i$$

$$w_2 = 2\left[\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right] = -1 - \sqrt{3}\,i$$



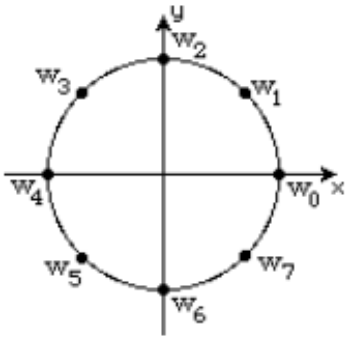
28. $(1)^{1/8} = \cos\dfrac{k\pi}{4} + i\sin\dfrac{k\pi}{4}, \quad k = 0,\ 1,\ 2,\ \ldots,\ 7$

$$w_0 = \cos 0 + i\sin 0 = 1; \quad w_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i$$

$$w_2 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i; \quad w_3 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i$$

$$w_4 = \cos\pi + i\sin\pi = -1; \quad w_5 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,i$$

$$w_6 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i; \quad w_7 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,i$$

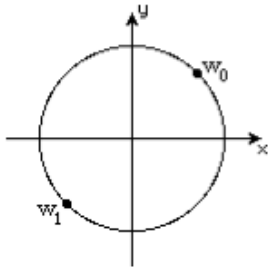


Soluções

29. $(i)^{1/2} = \cos\left(\frac{\pi}{4} + k\pi\right) + i\sin\left(\frac{\pi}{4} + k\pi\right), \quad k = 0, 1$

$$w_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i$$

$$w_1 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,i$$

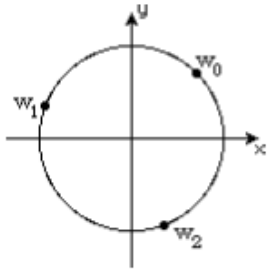


30. $(-1+i)^{1/3} = 2^{1/6}\left[\cos\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right)\right], \quad k = 0, 1, 2$

$$w_0 = 2^{1/6}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{2}}\,i = 0.7937 + 0.7937i$$

$$w_1 = 2^{1/6}\left[\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right] = -1.0842 + 0.2905i$$

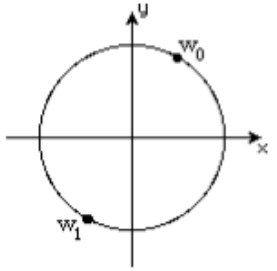
$$w_2 = 2^{1/6}\left[\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right] = 0.2905 - 1.0842i$$



31. $(-1+\sqrt{3}\,i)^{1/2} = 2^{1/2}\left[\cos\left(\frac{\pi}{3} + k\pi\right) + i\sin\left(\frac{\pi}{3} + k\pi\right)\right], \quad k = 0, 1$

$$w_0 = 2^{1/2}\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}\,i$$

$$w_2 = 2^{1/2}\left[\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right] = -\frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}\,i$$

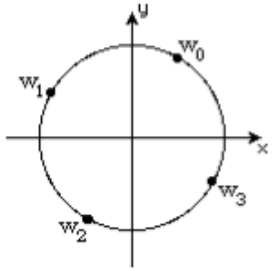


32. $(-1-\sqrt{3}\,i)^{1/4} = 2^{1/4}\left[\cos\left(\frac{\pi}{3} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{3} + \frac{k\pi}{2}\right)\right], \quad k = 0, 1, 2, 3$

$$w_0 = 2^{1/4}\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = 2^{1/4}\left[\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right]$$

$$w_1 = 2^{1/4}\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right] = 2^{1/4}\left[-\frac{\sqrt{3}}{2} + \frac{1}{2}\,i\right]$$

$$w_2 = 2^{1/4}\left[\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right] = 2^{1/4}\left[-\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right]; \quad w_3 = 2^{1/4}\left[\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right] = 2^{1/4}\left[\frac{\sqrt{3}}{2} - \frac{1}{2}\,i\right]$$



Soluções

33. The solutions are the four fourth roots of -1;

$$w_k = \cos\frac{\pi + 2k\pi}{4} + i\sin\frac{\pi + 2k\pi}{4}, \quad k = 0,\ 1,\ 2,\ 3.$$

We have

$$w_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i \qquad w_3 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,i$$

$$w_2 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i \qquad w_4 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,i.$$

34. $(z^4 - 1)^2 = 0$ is the same as $(z - i)^2(z + i)^2(z - 1)^2(z + 1)^2 = 0$. Thus $z_1 = 1$, $z_2 = -1$, $z_3 = i$, and $z_4 = -i$ are roots of multiplicity two.

35.
$$\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)^{12}\left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^5 = 2^5\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$
$$= 32\left[\cos\left(\frac{4\pi}{3} + \frac{5\pi}{6}\right) + i\sin\left(\frac{4\pi}{3} + \frac{5\pi}{6}\right)\right]$$
$$= 32\left(\cos\frac{13\pi}{6} + i\sin\frac{13\pi}{6}\right) = 32\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 16\sqrt{3} + 16i$$

36.
$$\frac{\left[8\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)\right]^3}{\left[2\left(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right)\right]^{10}} = \frac{2^9}{2^{10}}\left[\cos\left(\frac{9\pi}{8} - \frac{10\pi}{16}\right) + i\left(\frac{9\pi}{8} - \frac{10\pi}{16}\right)\right] = \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \frac{1}{2}\,i$$

37. We have

$$(\cos 2\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$

Also

$$(\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + (2\sin\theta\cos\theta)i.$$

Equating real and imaginary parts gives

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

Soluções

38. We have

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta.$$

Also

$$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 \\ &= \cos^3\theta - 3\cos\theta\sin^2\theta + (3\cos^2\theta\sin\theta - \sin^3\theta)i.\end{aligned}$$

Equating real and imaginary parts gives

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta, \quad \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

39. **(a)** $\operatorname{Arg}(z_1) = \pi$, $\operatorname{Arg}(z_2) = \dfrac{\pi}{2}$, $\operatorname{Arg}(z_1 z_2) = -\dfrac{\pi}{2}$, $\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) = \dfrac{3\pi}{2} \neq \operatorname{Arg}(z_1 z_2)$

(b) $\operatorname{Arg}(z_1/z_2) = -\dfrac{\pi}{2}$, $\operatorname{Arg}(z_1) - \operatorname{Arg}(z_2) = \pi - \dfrac{\pi}{2} = \dfrac{\pi}{2} \neq \operatorname{Arg}(z_1/z_2)$

40. **(a)** If we take $\arg(z_1) = \pi$ and $\arg(z_2) = \pi/2$ then $\arg(z_1) + \arg(z_2) = 3\pi/2$ is an argument of the product $z_1 z_2 = -5i$. With these same arguments we see that $\arg(z_1) - \arg(z_2) = \pi/2$ is an argument of the quotient $z_1/z_2 = \frac{1}{5}\,i$.

(b) If we take $\arg(z_1) = \pi$ and $\arg(z_2) = -\pi/2$ then $\arg(z_1) + \arg(z_2) = \pi/2$ is an argument of the product $z_1 z_2 = 5i$. With these same arguments we see that $\arg(z_1) - \arg(z_2) = 3\pi/2$ is an argument of the quotient $z_1/z_2 = -\frac{1}{5}\,i$.

Exercícios Funções Complexas Polinomiais: Gráficos e Limites

Nos Problemas 1-6, determine a imagem da reta indicada sob o mapeamento $f(z) = z^2$.

1. $y = 2$
2. $x = -3$
3. $x = 0$
4. $y = 0$
5. $y = x$
6. $y = -x$

Nos Problemas 7-14, expresse a função indicada na forma $f(z) = u + iv$.

7. $f(z) = 6z - 5 + 9i$
8. $f(z) = 7z - 9i\bar{z} - 3 + 2i$
9. $f(z) = z^2 - 3z + 4i$
10. $f(z) = 3\bar{z}^2 + 2z$
11. $f(z) = z^3 - 4z$
12. $f(z) = z^4$
13. $f(z) = z + 1/z$
14. $f(z) = \dfrac{z}{z+1}$

Nos Problemas 15-18, calcule a função dada nos pontos indicados.

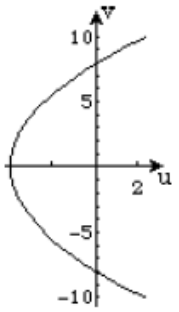
15. $f(z) = 2x - y^2 + i(xy^3 - 2x^2 + 1)$
    (a) $2i$ (b) $2 - i$ (c) $5 + 3i$
16. $f(z) = (x + 1 + 1/x) + i(4x^2 - 2y^2 - 4)$
    (a) $1 + i$ (b) $2 - i$ (c) $1 + 4i$
17. $f(z) = 4z + i\bar{z} + \mathrm{Re}(z)$
    (a) $4 - 6i$ (b) $-5 + 12i$ (c) $2 - 7i$
18. $f(z) = e^x \cos y + ie^x \operatorname{sen} y$
    (a) $\pi i/4$ (b) $-1 - \pi i$ (c) $3 + \pi i/3$

Nos Problemas 19-22, o limite indicado existe. Determine o seu valor.

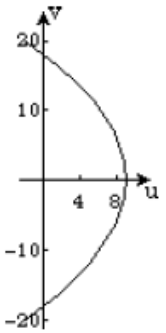
19. $\lim\limits_{z \to i}(4z^3 - 5z^2 + 4z + 1 - 5i)$
20. $\lim\limits_{z \to 1-i} \dfrac{5z^2 - 2z + 2}{z + 1}$

Soluções

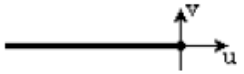
1. Substituting $y = 2$ into $u = x^2 - y^2$, $v = 2xy$ gives the parametric equations $u = x^2 - 4$, $v = 4x$. Using $x = v/4$ the first equation gives $u = v^2/16 - 4$. The graph is the parabola shown.



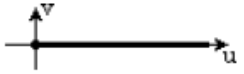
2. Substituting $x = -3$ into $u = x^2 - y^2$, $v = 2xy$ gives the parametric equations $u = 9 - y^2$, $v = -6y$. Using $y = -v/6$ the first equation gives $u = 9 - v^2/36$. The graph is the parabola shown.



3. $x = 0$ gives $u = -y^2$, $v = 0$. Since $-y^2 \leq 0$ for all real values of y, the image is the origin and the negative u-axis.



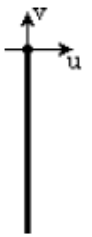
4. $y = 0$ gives $u = x^2$, $v = 0$. Since $x^2 \geq 0$ for all real values of x, the image is the origin and the positive u-axis.



5. $y = x$ gives $u = 0$, $v = 2x^2$. Since $x^2 \geq 0$ for all real values of x, the image is the origin and the positive v-axis.



6. $y = -x$ gives $u = 0$, $v = -2x^2$. Since $-x^2 \leq 0$ for all real values of x, the image is the origin and the negative v-axis.



7. $f(z) = (6x - 5) + i(6y + 9)$

8. $f(z) = (7x - 9y - 3) + i(7y - 9x + 2)$

9. $f(z) = (x^2 - y^2 - 3x) + i(2xy - 3y + 4)$

10. $f(z) = (3x^2 - 3y^2 + 2x) + i(-6xy + 2y)$

11. $f(z) = (x^3 - 3xy^2 - 4x) + i(3x^2y - y^3 - 4y)$

12. $f(z) = (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3)$

13. $f(z) = \left(x + \dfrac{x}{x^2 + y^2}\right) + i\left(y - \dfrac{y}{x^2 + y^2}\right)$

14. $f(z) = \dfrac{x^2 + y^2 + x}{(x+1)^2 + y^2} + i\,\dfrac{y}{(x+1)^2 + y^2}$

15. **(a)** $f(0 + 2i) = -4 + i$ **(b)** $f(2 - i) = 3 - 9i$ **(c)** $f(5 + 3i) = 1 + 86i$

16. **(a)** $f(1 + i) = 3 - 2i$ **(b)** $f(2 - i) = \dfrac{7}{2} + 10i$ **(c)** $f(1 + 4i) = 3 - 32i$

17. **(a)** $f(4 - 6i) = 14 - 20i$ **(b)** $f(-5 + 12i) = -13 + 43i$ **(c)** $f(2 - 7i) = 3 - 26i$

18. **(a)** $f(0 + \dfrac{\pi}{4}\,i) = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}\,i$ **(b)** $f(-1 - \pi i) = -e^{-1}$ **(c)** $f(3 + \dfrac{\pi}{3}\,i) = \dfrac{1}{2}e^3 + \dfrac{\sqrt{3}}{2}\,e^3 i$

19. $\lim\limits_{z \to i}(4z^3 - 5z^2 + 4z + 1 - 5i) = 6 - 5i$

20. $\lim\limits_{z \to 1-i} \dfrac{5z^2 - 2z + 2}{z + 1} = \dfrac{5(1 - i)^2 - 2(1 - i) + 2}{2 - i} = \dfrac{8}{5} - \dfrac{16}{5}\,i$

Exercícios: Funções Complexas Exponenciais

Nos Problemas 1-10, expressar e^z na forma $a + ib$.

1. $z = \frac{\pi}{6} i$
2. $z = -\frac{\pi}{3} i$
3. $z = -1 + \frac{\pi}{4} i$
4. $z = 2 - \frac{\pi}{2} i$
5. $z = \pi + \pi i$
6. $z = -\pi + \frac{3\pi}{2} i$
7. $z = 1{,}5 + 2i$
8. $z = -0{,}3 + 0{,}5i$
9. $z = 5i$
10. $z = -0{,}23 - i$

Nos Problemas 11 e 12, expresse o número indicado na forma $a + ib$.

11. $e^{1+5\pi i/4} e^{-1-\pi i/3}$
12. $\dfrac{e^{2+3\pi i}}{e^{-3+\pi i/2}}$

Nos Problemas 13-16, use a Definição 8.9 para expressar a função indicada na forma $f(z) = u + iv$.

13. $f(z) = e^{-iz}$
14. $f(z) = e^{2\bar{z}}$
15. $f(z) = e^{z^2}$
16. $f(z) = e^{1/z}$

Nos Problemas 17-20, verifique o resultado indicado.

17. $|e^z| = e^x$
18. $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$
19. $e^{z+\pi i} = e^{z-\pi i}$
20. $(e^z)^n = e^{nz}$, n um inteiro

Soluções

1. $e^{\frac{\pi}{6}i} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}\,i$
2. $e^{-\frac{\pi}{3}i} = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}\,i$
3. $e^{-1+\frac{\pi}{4}i} = e^{-1}\cos\frac{\pi}{4} + ie^{-1}\sin\frac{\pi}{4} = e^{-1}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i\right)$
4. $e^{2-\frac{\pi}{2}i} = e^2\cos\left(-\frac{\pi}{2}\right) + ie^2\sin\left(-\frac{\pi}{2}\right) = -e^2 i$
5. $e^{\pi+\pi i} = e^{\pi}\cos\pi + ie^{\pi}\sin\pi = -e^{\pi}$
6. $e^{-\pi+\frac{3\pi}{2}i} = e^{-\pi}\cos\frac{3\pi}{2} + ie^{-\pi}\sin\frac{3\pi}{2} = -e^{-\pi}i$
7. $e^{1.5+2i} = e^{1.5}\cos 2 + ie^{1.5}\sin 2 = -1.8650 + 4.0752i$
8. $e^{-0.3+0.5i} = e^{-0.3}\cos 0.5 + ie^{-0.3}\sin 0.5 = 0.6501 + 0.3552i$
9. $e^{5i} = \cos 5 + i\sin 5 = 0.2837 - 0.9589i$
10. $e^{-0.23-i} = e^{-0.23}\cos(-1) + ie^{-0.23}\sin(-1) = 0.4293 - 0.6686i$
11. $e^{\frac{11\pi}{12}i} = \cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12} = -0.9659 + 0.2588i$
12. $e^{5+\frac{5\pi}{2}i} = e^5\cos\frac{5\pi}{2} + ie^5\sin\frac{5\pi}{2} = e^5 i$
13. $e^{-iz} = e^{y-xi} = e^y\cos x - ie^y\sin x$
14. $e^{2\bar{z}} = e^{2x-2yi} = e^{2x}\cos 2y - ie^{2x}\sin 2y$
15. $e^{z^2} = e^{x^2-y^2+2xyi} = e^{x^2-y^2}\cos 2xy + ie^{x^2-y^2}\sin 2xy$
16. $e^{1/z} = e^{x/(x^2+y^2)-iy/(x^2+y^2)} = e^{x/(x^2+y^2)}\cos\frac{y}{x^2+y^2} - ie^{x/(x^2+y^2)}\sin\frac{y}{x^2+y^2}$

Soluções

17. $|e^z|^2 = e^{2x}\cos^2 y + e^{2x}\sin^2 y = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x}$ implies $|e^z| = e^x$.

18. $$\frac{e^{z_1}}{e^{z_2}} = \frac{e^{x_1}\cos y_1 + ie^{x_1}\sin y_1}{e^{x_2}\cos y_2 + ie^{x_2}\sin y_2} = \frac{(e^{x_1}\cos y_1 + ie^{x_1}\sin y_1)(e^{x_2}\cos y_2 - ie^{x_2}\sin y_2)}{e^{2x_2}}$$

$$= e^{x_1-x_2}[(\cos y_1\cos y_2 + \sin y_1\sin y_2) + i(\sin y_1\cos y_2 - \cos y_1\sin y_2)]$$

$$= e^{x_1-x_2}[\cos(y_1-y_2) + i\sin(y_1-y_2)] = e^{x_1-x_2+i(y_1-y_2)} = e^{(x_1+iy_1)-(x_2+iy_2)} = e^{z_1-z_2}$$

19. $e^{z+\pi i} = e^{x+(y+\pi)i} = e^x[\cos(y+\pi) + i\sin(y+\pi)] = e^x[\cos(y-\pi) + i\sin(y-\pi)] = e^{x+(y-\pi)i} = e^{z-\pi i}$

20. $(e^z)^n = (e^x[\cos y + i\sin y])^n = e^{nx}[\cos y + i\sin y]^n = e^{nx}[\cos ny + i\sin ny] = e^{nz}$, n an integer

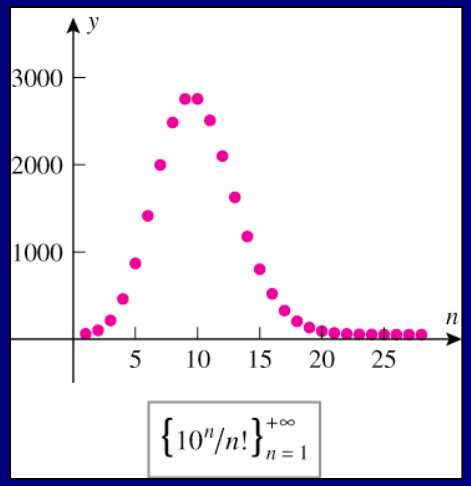
Exercícios: Funções Complexas Trigonométricas

Nos Problemas 1- 8, expresse a quantidade indicada na forma $a + ib$.

1. $\cos(3i)$
2. $\operatorname{sen}(-2i)$
3. $\operatorname{sen}\left(\frac{\pi}{4} + i\right)$
4. $\cos(2 - 4i)$
5. $\operatorname{tg}(i)$
6. $\operatorname{cotg}\left(\frac{\pi}{2} + 3i\right)$
7. $\sec(\pi + i)$
8. $\operatorname{cosec}(1 + i)$

Soluções

1. $\cos(3i) = \cosh 3 = 10.0677$
2. $\sin(-2i) = i\sinh(-2) = -3.6269i$
3. $\sin\left(\frac{\pi}{4} + i\right) = \sin\frac{\pi}{4}\cosh(1) + i\cos\frac{\pi}{4}\sinh(1) = 1.0911 + 0.8310i$
4. $\cos(2 - 4i) = \cos(2)\cosh(-4) - \sin(2)\sinh(-4) = -11.3642 - 24.8147i$
5. $\tan(i) = \frac{\sin(i)}{\cos(i)} = \frac{i\sinh(1)}{\cosh(1)} = 0.7616i$
6. $\cot\left(\frac{\pi}{2} + 3i\right) = \frac{\cos(\frac{\pi}{2} + 3i)}{\sin(\frac{\pi}{2} + 3i)} = \frac{-i\sinh(3)}{\cosh(3)} = -0.9951i$
7. $\sec(\pi + i) = \frac{1}{\cos(\pi + i)} = \frac{1}{-\cosh(1)} = -0.6481$
8. $\csc(1 + i) = \frac{1}{\sin(1 + i)} = \frac{1}{\sin(1)\cosh(1) + i\cos(1)\sinh(1)} = 0.6215 - 0.3039i$



IV– Sequências e Séries

Obra de Referência: *Cálculo*, vol 2, Howard Anton

Exercícios: Determinação de Sequencias e Limites

1. Em cada parte, encontre a fórmula para o termo geral da seqüência, começando com $n = 1$.

(a) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ (b) $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots$

(c) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$ (d) $\frac{1}{\sqrt{\pi}}, \frac{4}{\sqrt[3]{\pi}}, \frac{9}{\sqrt[4]{\pi}}, \frac{16}{\sqrt[5]{\pi}}, \ldots$

2. Em cada parte, encontre duas fórmulas para o termo geral da seqüência, uma começando com $n = 1$ e a outra com $n = 0$.

(a) $1, -r, r^2, -r^3, \ldots$ (b) $r, -r^2, r^3, -r^4, \ldots$

3. (a) Escreva os quatro primeiros termos da seqüência $\{1+(-1)^n\}$, começando com $n = 0$.

(b) Escreva os quatro primeiros termos da seqüência $\{\cos n\pi\}$, começando com $n = 0$.

(c) Use os resultados nas partes (a) e (b) para expressar o termo geral da seqüência 4, 0, 4, 0,... de duas formas diferentes, começando com $n = 0$.

4. Em cada parte, encontre a fórmula para o termo geral usando fatoriais e começando com $n = 1$.

(a) $1 \cdot 2, 1 \cdot 2 \cdot 3 \cdot 4, 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6,$
$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8, \ldots$

(b) $1, 1 \cdot 2 \cdot 3, 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5, 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7, \ldots$

5-22 Escreva os cinco primeiros termos da seqüência, determine se ela converge e, se isso acontecer, encontre o limite.

5. $\left\{\frac{n}{n+2}\right\}_{n=1}^{+\infty}$ **6.** $\left\{\frac{n^2}{2n+1}\right\}_{n=1}^{+\infty}$ **7.** $\{2\}_{n=1}^{+\infty}$

8. $\left\{\ln\left(\frac{1}{n}\right)\right\}_{n=1}^{+\infty}$ **9.** $\left\{\frac{\ln n}{n}\right\}_{n=1}^{+\infty}$ **10.** $\left\{n \operatorname{sen}\frac{\pi}{n}\right\}_{n=1}^{+\infty}$

11. $\{1 + (-1)^n\}_{n=1}^{+\infty}$ **12.** $\left\{\frac{(-1)^{n+1}}{n^2}\right\}_{n=1}^{+\infty}$

13. $\left\{(-1)^n \dfrac{2n^3}{n^3+1}\right\}_{n=1}^{+\infty}$

14. $\left\{\dfrac{n}{2^n}\right\}_{n=1}^{+\infty}$

15. $\left\{\dfrac{(n+1)(n+2)}{2n^2}\right\}_{n=1}^{+\infty}$

16. $\left\{\dfrac{\pi^n}{4^n}\right\}_{n=1}^{+\infty}$

17. $\left\{\cos\dfrac{3}{n}\right\}_{n=1}^{+\infty}$

18. $\left\{\cos\dfrac{\pi n}{2}\right\}_{n=1}^{+\infty}$

19. $\{n^2 e^{-n}\}_{n=1}^{+\infty}$

20. $\{\sqrt{n^2+3n}-n\}_{n=1}^{+\infty}$

21. $\left\{\left(\dfrac{n+3}{n+1}\right)^n\right\}_{n=1}^{+\infty}$

22. $\left\{\left(1-\dfrac{2}{n}\right)^n\right\}_{n=1}^{+\infty}$

Soluções

1. (a) $\dfrac{1}{3^{n-1}}$ (b) $\dfrac{(-1)^{n-1}}{3^{n-1}}$ (c) $\dfrac{2n-1}{2n}$ (d) $\dfrac{n^2}{\pi^{1/(n+1)}}$

2. (a) $(-r)^{n-1}$; $(-r)^n$ (b) $(-1)^{n+1}r^n$; $(-1)^n r^{n+1}$

3. (a) $2, 0, 2, 0$ (b) $1, -1, 1, -1$ (c) $2(1+(-1)^n)$; $2+2\cos n\pi$

4. (a) $(2n)!$ (b) $(2n-1)!$

5. $1/3,\ 2/4,\ 3/5,\ 4/6,\ 5/7, \ldots$; $\lim\limits_{n\to+\infty} \dfrac{n}{n+2} = 1$, converges

6. $1/3,\ 4/5,\ 9/7,\ 16/9,\ 25/11, \ldots$; $\lim\limits_{n\to+\infty} \dfrac{n^2}{2n+1} = +\infty$, diverges

7. $2, 2, 2, 2, 2, \ldots$; $\lim\limits_{n\to+\infty} 2 = 2$, converges

8. $\ln 1,\ \ln\dfrac{1}{2},\ \ln\dfrac{1}{3},\ \ln\dfrac{1}{4},\ \ln\dfrac{1}{5}, \ldots$; $\lim\limits_{n\to+\infty} \ln(1/n) = -\infty$, diverges

9. $\dfrac{\ln 1}{1}, \dfrac{\ln 2}{2}, \dfrac{\ln 3}{3}, \dfrac{\ln 4}{4}, \dfrac{\ln 5}{5}, \ldots$;

$\lim\limits_{n\to+\infty} \dfrac{\ln n}{n} = \lim\limits_{n\to+\infty} \dfrac{1}{n} = 0$ $\left(\text{apply L'Hôpital's Rule to } \dfrac{\ln x}{x}\right)$, converges

10. $\sin\pi,\ 2\sin(\pi/2),\ 3\sin(\pi/3),\ 4\sin(\pi/4),\ 5\sin(\pi/5), \ldots$;

$\lim\limits_{n\to+\infty} n\sin(\pi/n) = \lim\limits_{n\to+\infty} \dfrac{\sin(\pi/n)}{1/n} = \lim\limits_{n\to+\infty} \dfrac{(-\pi/n^2)\cos(\pi/n)}{-1/n^2} = \pi$, converges

11. $0, 2, 0, 2, 0, \ldots$; diverges

12. $1,\ -1/4,\ 1/9,\ -1/16,\ 1/25, \ldots$; $\lim\limits_{n\to+\infty} \dfrac{(-1)^{n+1}}{n^2} = 0$, converges

Soluções

13. $-1,\ 16/9,\ -54/28,\ 128/65,\ -250/126,\ldots$; diverges because odd-numbered terms approach -2, even-numbered terms approach 2.

14. $1/2,\ 2/4,\ 3/8,\ 4/16,\ 5/32,\ldots$; $\lim_{n\to+\infty} \dfrac{n}{2^n} = \lim_{n\to+\infty} \dfrac{1}{2^n \ln 2} = 0$, converges

15. $6/2,\ 12/8,\ 20/18,\ 30/32,\ 42/50,\ldots$; $\lim_{n\to+\infty} \dfrac{1}{2}(1+1/n)(1+2/n) = 1/2$, converges

16. $\pi/4,\ \pi^2/4^2,\ \pi^3/4^3,\ \pi^4/4^4,\ \pi^5/4^5,\ldots$; $\lim_{n\to+\infty} (\pi/4)^n = 0$, converges

17. $\cos(3),\ \cos(3/2),\ \cos(1),\ \cos(3/4),\ \cos(3/5),\ldots$; $\lim_{n\to+\infty} \cos(3/n) = 1$, converges

18. $0,\ -1,\ 0,\ 1,\ 0,\ldots$; diverges

19. $e^{-1},\ 4e^{-2},\ 9e^{-3},\ 16e^{-4},\ 25e^{-5},\ldots$; $\lim_{x\to+\infty} x^2 e^{-x} = \lim_{x\to+\infty} \dfrac{x^2}{e^x} = 0$, so $\lim_{n\to+\infty} n^2 e^{-n} = 0$, converges

20. $1,\ \sqrt{10}-2,\ \sqrt{18}-3,\ \sqrt{28}-4,\ \sqrt{40}-5,\ldots$;

$$\lim_{n\to+\infty} (\sqrt{n^2+3n}-n) = \lim_{n\to+\infty} \frac{3n}{\sqrt{n^2+3n}+n} = \lim_{n\to+\infty} \frac{3}{\sqrt{1+3/n}+1} = \frac{3}{2}, \text{ converges}$$

21. $2,\ (5/3)^2,\ (6/4)^3,\ (7/5)^4,\ (8/6)^5,\ldots$; let $y = \left[\dfrac{x+3}{x+1}\right]^x$, converges because

$$\lim_{x\to+\infty} \ln y = \lim_{x\to+\infty} \frac{\ln \dfrac{x+3}{x+1}}{1/x} = \lim_{x\to+\infty} \frac{2x^2}{(x+1)(x+3)} = 2, \text{ so } \lim_{n\to+\infty} \left[\frac{n+3}{n+1}\right]^n = e^2$$

22. $-1,\ 0,\ (1/3)^3,\ (2/4)^4,\ (3/5)^5,\ldots$; let $y = (1-2/x)^x$, converges because

$$\lim_{x\to+\infty} \ln y = \lim_{x\to+\infty} \frac{\ln(1-2/x)}{1/x} = \lim_{x\to+\infty} \frac{-2}{1-2/x} = -2,\ \lim_{n\to+\infty} (1-2/n)^n = \lim_{x\to+\infty} y = e^{-2}$$

Exercícios: Verificação de Monotonicidade

1-6 Use $a_{n+1} - a_n$ para mostrar que a seqüência $\{a_n\}$ dada é estritamente crescente ou estritamente decrescente.

1. $\left\{\frac{1}{n}\right\}_{n=1}^{+\infty}$ **2.** $\left\{1 - \frac{1}{n}\right\}_{n=1}^{+\infty}$ **3.** $\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$

4. $\left\{\frac{n}{4n-1}\right\}_{n=1}^{+\infty}$ **5.** $\{n - 2^n\}_{n=1}^{+\infty}$ **6.** $\{n - n^2\}_{n=1}^{+\infty}$

7-12 Use a_{n+1}/a_n para mostrar que a seqüência $\{a_n\}$ dada é estritamente crescente ou estritamente decrescente.

7. $\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$ **8.** $\left\{\frac{2^n}{1+2^n}\right\}_{n=1}^{+\infty}$ **9.** $\{ne^{-n}\}_{n=1}^{+\infty}$

10. $\left\{\frac{10^n}{(2n)!}\right\}_{n=1}^{+\infty}$ **11.** $\left\{\frac{n^n}{n!}\right\}_{n=1}^{+\infty}$ **12.** $\left\{\frac{5^n}{2^{(n^2)}}\right\}_{n=1}^{+\infty}$

1. $a_{n+1} - a_n = \dfrac{1}{n+1} - \dfrac{1}{n} = -\dfrac{1}{n(n+1)} < 0$ for $n \geq 1$, so strictly decreasing.

2. $a_{n+1} - a_n = (1 - \dfrac{1}{n+1}) - (1 - \dfrac{1}{n}) = \dfrac{1}{n(n+1)} > 0$ for $n \geq 1$, so strictly increasing.

3. $a_{n+1} - a_n = \dfrac{n+1}{2n+3} - \dfrac{n}{2n+1} = \dfrac{1}{(2n+1)(2n+3)} > 0$ for $n \geq 1$, so strictly increasing.

4. $a_{n+1} - a_n = \dfrac{n+1}{4n+3} - \dfrac{n}{4n-1} = -\dfrac{1}{(4n-1)(4n+3)} < 0$ for $n \geq 1$, so strictly decreasing.

5. $a_{n+1} - a_n = (n+1-2^{n+1}) - (n-2^n) = 1 - 2^n < 0$ for $n \geq 1$, so strictly decreasing.

6. $a_{n+1} - a_n = [(n+1) - (n+1)^2] - (n-n^2) = -2n < 0$ for $n \geq 1$, so strictly decreasing.

7. $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)/(2n+3)}{n/(2n+1)} = \dfrac{(n+1)(2n+1)}{n(2n+3)} = \dfrac{2n^2+3n+1}{2n^2+3n} > 1$ for $n \geq 1$, so strictly increasing.

8. $\dfrac{a_{n+1}}{a_n} = \dfrac{2^{n+1}}{1+2^{n+1}} \cdot \dfrac{1+2^n}{2^n} = \dfrac{2+2^{n+1}}{1+2^{n+1}} = 1 + \dfrac{1}{1+2^{n+1}} > 1$ for $n \geq 1$, so strictly increasing.

9. $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)e^{-(n+1)}}{ne^{-n}} = (1+1/n)e^{-1} < 1$ for $n \geq 1$, so strictly decreasing.

10. $\dfrac{a_{n+1}}{a_n} = \dfrac{10^{n+1}}{(2n+2)!} \cdot \dfrac{(2n)!}{10^n} = \dfrac{10}{(2n+2)(2n+1)} < 1$ for $n \geq 1$, so strictly decreasing.

11. $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)^{n+1}}{(n+1)!} \cdot \dfrac{n!}{n^n} = \dfrac{(n+1)^n}{n^n} = (1+1/n)^n > 1$ for $n \geq 1$, so strictly increasing.

12. $\dfrac{a_{n+1}}{a_n} = \dfrac{5^{n+1}}{2^{(n+1)^2}} \cdot \dfrac{2^{n^2}}{5^n} = \dfrac{5}{2^{2n+1}} < 1$ for $n \geq 1$, so strictly decreasing.

Exercícios: Verificação de Monotonicidade

13-18 Use diferenciação para mostrar que a seqüência dada é estritamente crescente ou estritamente decrescente.

13. $\left\{\dfrac{n}{2n+1}\right\}_{n=1}^{+\infty}$

14. $\left\{3-\dfrac{1}{n}\right\}_{n=1}^{+\infty}$

15. $\left\{\dfrac{1}{n+\ln n}\right\}_{n=1}^{+\infty}$

16. $\{ne^{-2n}\}_{n=1}^{+\infty}$

17. $\left\{\dfrac{\ln(n+2)}{n+2}\right\}_{n=1}^{+\infty}$

18. $\{\text{arc tg } n\}_{n=1}^{+\infty}$

19-24 Mostre que a seqüência dada é, a partir de um certo termo, estritamente crescente ou estritamente decrescente.

19. $\{2n^2-7n\}_{n=1}^{+\infty}$

20. $\{n^3-4n^2\}_{n=1}^{+\infty}$

21. $\left\{\dfrac{n}{n^2+10}\right\}_{n=1}^{+\infty}$

22. $\left\{n+\dfrac{17}{n}\right\}_{n=1}^{+\infty}$

23. $\left\{\dfrac{n!}{3^n}\right\}_{n=1}^{+\infty}$

24. $\{n^5e^{-n}\}_{n=1}^{+\infty}$

13. $f(x) = x/(2x+1)$, $f'(x) = 1/(2x+1)^2 > 0$ for $x \geq 1$, so strictly increasing.

14. $f(x) = 3 - 1/x$, $f'(x) = 1/x^2 > 0$ for $x \geq 1$, so strictly increasing.

15. $f(x) = 1/(x + \ln x)$, $f'(x) = -\dfrac{1 + 1/x}{(x + \ln x)^2} < 0$ for $x \geq 1$, so strictly decreasing.

16. $f(x) = xe^{-2x}$, $f'(x) = (1 - 2x)e^{-2x} < 0$ for $x \geq 1$, so strictly decreasing.

17. $f(x) = \dfrac{\ln(x+2)}{x+2}$, $f'(x) = \dfrac{1 - \ln(x+2)}{(x+2)^2} < 0$ for $x \geq 1$, so strictly decreasing.

18. $f(x) = \tan^{-1} x$, $f'(x) = 1/(1 + x^2) > 0$ for $x \geq 1$, so strictly increasing.

19. $f(x) = 2x^2 - 7x$, $f'(x) = 4x - 7 > 0$ for $x \geq 2$, so eventually strictly increasing.

20. $f(x) = x^3 - 4x^2$, $f'(x) = 3x^2 - 8x = x(3x - 8) > 0$ for $x \geq 3$, so eventually strictly increasing.

21. $f(x) = \dfrac{x}{x^2 + 10}$, $f'(x) = \dfrac{10 - x^2}{(x^2 + 10)^2} < 0$ for $x \geq 4$, so eventually strictly decreasing.

22. $f(x) = x + \dfrac{17}{x}$, $f'(x) = \dfrac{x^2 - 17}{x^2} > 0$ for $x \geq 5$, so eventually strictly increasing.

23. $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!}{3^{n+1}} \cdot \dfrac{3^n}{n!} = \dfrac{n+1}{3} > 1$ for $n \geq 3$, so eventually strictly increasing.

24. $f(x) = x^5 e^{-x}$, $f'(x) = x^4(5 - x)e^{-x} < 0$ for $x \geq 6$, so eventually strictly decreasing.

Exercícios: Séries - Verificação de Convergência

3-14 Determine se a série converge e, se convergir, encontre sua soma.

3. $\displaystyle\sum_{k=1}^{\infty}\left(-\frac{3}{4}\right)^{k-1}$

4. $\displaystyle\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k+2}$

5. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k-1}\frac{7}{6^{k-1}}$

6. $\displaystyle\sum_{k=1}^{\infty}\left(-\frac{3}{2}\right)^{k+1}$

7. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(k+2)(k+3)}$

8. $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{2^k}-\frac{1}{2^{k+1}}\right)$

9. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{9k^2+3k-2}$

10. $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k^2-1}$

11. $\displaystyle\sum_{k=3}^{\infty}\frac{1}{k-2}$

12. $\displaystyle\sum_{k=5}^{\infty}\left(\frac{e}{\pi}\right)^{k-1}$

13. $\displaystyle\sum_{k=1}^{\infty}\frac{4^{k+2}}{7^{k-1}}$

14. $\displaystyle\sum_{k=1}^{\infty}5^{3k}7^{1-k}$

3. geometric, $a = 1$, $r = -3/4$, sum $= \dfrac{1}{1-(-3/4)} = 4/7$

4. geometric, $a = (2/3)^3$, $r = 2/3$, sum $= \dfrac{(2/3)^3}{1-2/3} = 8/9$

5. geometric, $a = 7$, $r = -1/6$, sum $= \dfrac{7}{1+1/6} = 6$

6. geometric, $r = -3/2$, diverges

7. $s_n = \displaystyle\sum_{k=1}^{n}\left(\frac{1}{k+2} - \frac{1}{k+3}\right) = \frac{1}{3} - \frac{1}{n+3}$, $\lim\limits_{n\to+\infty} s_n = 1/3$

8. $s_n = \displaystyle\sum_{k=1}^{n}\left(\frac{1}{2^k} - \frac{1}{2^{k+1}}\right) = \frac{1}{2} - \frac{1}{2^{n+1}}$, $\lim\limits_{n\to+\infty} s_n = 1/2$

9. $s_n = \displaystyle\sum_{k=1}^{n}\left(\frac{1/3}{3k-1} - \frac{1/3}{3k+2}\right) = \frac{1}{6} - \frac{1/3}{3n+2}$, $\lim\limits_{n\to+\infty} s_n = 1/6$

10. $s_n = \displaystyle\sum_{k=2}^{n+1}\left[\frac{1/2}{k-1} - \frac{1/2}{k+1}\right] = \frac{1}{2}\left[\sum_{k=2}^{n+1}\frac{1}{k-1} - \sum_{k=2}^{n+1}\frac{1}{k+1}\right]$

$= \dfrac{1}{2}\left[\displaystyle\sum_{k=2}^{n+1}\frac{1}{k-1} - \sum_{k=4}^{n+3}\frac{1}{k-1}\right] = \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]$; $\lim\limits_{n\to+\infty} s_n = \dfrac{3}{4}$

11. $\displaystyle\sum_{k=3}^{\infty}\frac{1}{k-2} = \sum_{k=1}^{\infty} 1/k$, the harmonic series, so the series diverges.

12. geometric, $a = (e/\pi)^4$, $r = e/\pi < 1$, sum $= \dfrac{(e/\pi)^4}{1-e/\pi} = \dfrac{e^4}{\pi^3(\pi-e)}$

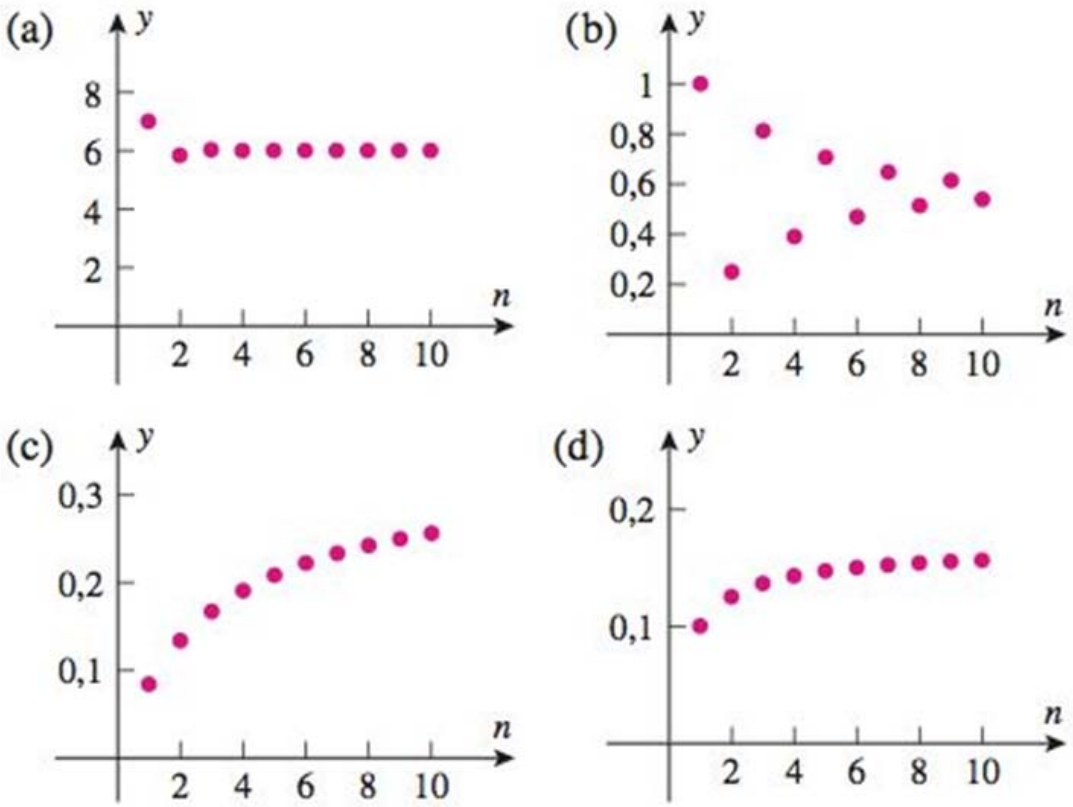
13. $\displaystyle\sum_{k=1}^{\infty}\frac{4^{k+2}}{7^{k-1}} = \sum_{k=1}^{\infty} 64\left(\frac{4}{7}\right)^{k-1}$; geometric, $a = 64$, $r = 4/7$, sum $= \dfrac{64}{1-4/7} = 448/3$

14. geometric, $a = 125, r = 125/7$, diverges

Exercícios:

Séries x Sequencias

15. Associe uma série de um dos Exercícios 3, 5, 7 ou 9 com o gráfico da seqüência de suas somas parciais.



17-20 Expresse a dízima periódica como uma fração.

17. 0,4444 …

18. 0,9999 …

19. 5,373737 …

20. 0,451141414 …

15. **(a)** Exercise 5 **(b)** Exercise 3 **(c)** Exercise 7 **(d)** Exercise 9

17. $0.4444\cdots = 0.4 + 0.04 + 0.004 + \cdots = \dfrac{0.4}{1 - 0.1} = 4/9$

18. $0.9999\cdots = 0.9 + 0.09 + 0.009 + \cdots = \dfrac{0.9}{1 - 0.1} = 1$

19. $5.373737\cdots = 5 + 0.37 + 0.0037 + 0.000037 + \cdots = 5 + \dfrac{0.37}{1 - 0.01} = 5 + 37/99 = 532/99$

20. $0.451141414\cdots = 0.451 + 0.00014 + 0.0000014 + 0.000000014 + \cdots = 0.451 + \dfrac{0.00014}{1 - 0.01} = \dfrac{44663}{99000}$

Exercícios: Séries de Taylor e MacLaurin

1-10 Use a notação de somatório para escrever a série de Maclaurin para a função dada.

1. e^{-x} **2.** e^{ax} **3.** $\cos \pi x$ **4.** $\operatorname{sen} \pi x$

5. $\ln(1+x)$ **6.** $\dfrac{1}{1+x}$ **7.** $\cosh x$

8. $\operatorname{senh} x$ **9.** $x \operatorname{sen} x$ **10.** xe^{x}

11-18 Use a notação de somatório para escrever a série de Taylor em torno de $x = x_0$ para a função dada.

11. $e^{x};\ x_0 = 1$ **12.** $e^{-x};\ x_0 = \ln 2$

13. $\dfrac{1}{x};\ x_0 = -1$ **14.** $\dfrac{1}{x+2};\ x_0 = 3$

15. $\operatorname{sen} \pi x;\ x_0 = \dfrac{1}{2}$ **16.** $\cos x;\ x_0 = \dfrac{\pi}{2}$

17. $\ln x;\ x_0 = 1$ **18.** $\ln x;\ x_0 = e$

1. $f^{(k)}(x) = (-1)^k e^{-x}$, $f^{(k)}(0) = (-1)^k$; $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k$

2. $f^{(k)}(x) = a^k e^{ax}$, $f^{(k)}(0) = a^k$; $\displaystyle\sum_{k=0}^{\infty} \frac{a^k}{k!} x^k$

3. $f^{(k)}(0) = 0$ if k is odd, $f^{(k)}(0)$ is alternately π^k and $-\pi^k$ if k is even; $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k}}{(2k)!} x^{2k}$

4. $f^{(k)}(0) = 0$ if k is even, $f^{(k)}(0)$ is alternately π^k and $-\pi^k$ if k is odd; $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1}}{(2k+1)!} x^{2k+1}$

5. $f^{(0)}(0) = 0$; for $k \geq 1$, $f^{(k)}(x) = \dfrac{(-1)^{k+1}(k-1)!}{(1+x)^k}$, $f^{(k)}(0) = (-1)^{k+1}(k-1)!$; $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$

6. $f^{(k)}(x) = (-1)^k \dfrac{k!}{(1+x)^{k+1}}$; $f^{(k)}(0) = (-1)^k k!$; $\displaystyle\sum_{k=0}^{\infty} (-1)^k x^k$

7. $f^{(k)}(0) = 0$ if k is odd, $f^{(k)}(0) = 1$ if k is even; $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k}$

8. $f^{(k)}(0) = 0$ if k is even, $f^{(k)}(0) = 1$ if k is odd; $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1}$

9. $f^{(k)}(x) = \begin{cases} (-1)^{k/2}(x \sin x - k \cos x) & k \text{ even} \\ (-1)^{(k-1)/2}(x \cos x + k \sin x) & k \text{ odd} \end{cases}$, $f^{(k)}(0) = \begin{cases} (-1)^{1+k/2} k & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$

$\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+2}$

10. $f^{(k)}(x) = (k+x)e^x$, $f^{(k)}(0) = k$; $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^k$

11. $f^{(k)}(x_0) = e$; $\displaystyle\sum_{k=0}^{\infty} \frac{e}{k!}(x-1)^k$

12. $f^{(k)}(x) = (-1)^k e^{-x}$, $f^{(k)}(\ln 2) = (-1)^k \frac{1}{2}$; $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2 \cdot k!}(x - \ln 2)^k$

13. $f^{(k)}(x) = \dfrac{(-1)^k k!}{x^{k+1}}$, $f^{(k)}(-1) = -k!$; $\displaystyle\sum_{k=0}^{\infty} (-1)(x+1)^k$

14. $f^{(k)}(x) = \dfrac{(-1)^k k!}{(x+2)^{k+1}}$, $f^{(k)}(3) = \dfrac{(-1)^k k!}{5^{k+1}}$; $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{5^{k+1}}(x-3)^k$

15. $f^{(k)}(1/2) = 0$ if k is odd, $f^{(k)}(1/2)$ is alternately π^k and $-\pi^k$ if k is even;

$\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k}}{(2k)!}(x - 1/2)^{2k}$

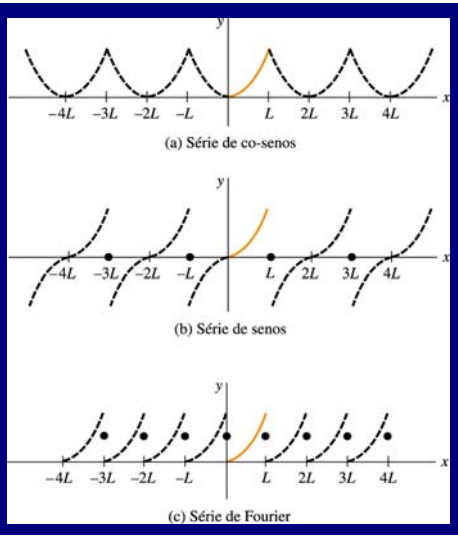
16. $f^{(k)}(\pi/2) = 0$ if k is even, $f^{(k)}(\pi/2)$ is alternately -1 and 1 if k is odd; $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!}(x - \pi/2)^{2k+1}$

17. $f(1) = 0$, for $k \geq 1$, $f^{(k)}(x) = \dfrac{(-1)^{k-1}(k-1)!}{x^k}$; $f^{(k)}(1) = (-1)^{k-1}(k-1)!$;

$\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-1)^k$

18. $f(e) = 1$, for $k \geq 1$, $f^{(k)}(x) = \dfrac{(-1)^{k-1}(k-1)!}{x^k}$; $f^{(k)}(e) = \dfrac{(-1)^{k-1}(k-1)!}{e^k}$;

$1 + \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k e^k}(x-e)^k$



VI – A Análise de Fourier

Obra de referência: Matemática Avançada para Engenharia, Dennis Zill (Vols 2 e 3)

Exercícios: Espaços Vetoriais (Zill, vol 2)

Nos Problemas 1-10, determine se o conjunto indicado é um espaço vetorial. Se não for, apresente pelo menos um axioma que não seja satisfeito. A menos que seja dito o contrário, considere que a adição vetorial e a multiplicação escalar são as operações ordinárias definidas no conjunto.

1. O conjunto de vetores $\langle a_1, a_2\rangle$, onde $a_1 \geq 0, a_2 \geq 0$
2. O conjunto de vetores $\langle a_1, a_2\rangle$, onde $a_2 = 3a_1 + 1$
3. O conjunto de vetores $\langle a_1, a_2\rangle$, multiplicação escalar definida por $k\langle a_1, a_2\rangle = \langle ka_1, 0\rangle$
4. O conjunto de vetores $\langle a_1, a_2\rangle$, onde $a_1 + a_2 = 0$
5. O conjunto de vetores $\langle a_1, a_2, 0\rangle$
6. O conjunto de vetores $\langle a_1, a_2\rangle$, adição e multiplicação escalar definidas por

$$\langle a_1, a_2\rangle + \langle b_1, b_2\rangle = \langle a_1 + b_1 + 1, a_2 + b_2 + 1\rangle$$
$$k\langle a_1, a_2\rangle = \langle ka_1 + k - 1, ka_2 + k - 1\rangle$$

7. O conjunto de números reais, adição definida por $\mathbf{x} + \mathbf{y} = x - y$
8. O conjunto de números complexos $a + bi$, onde $i^2 = -1$, adição e multiplicação escalar definidas por

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$$
$$k(a + bi) = ka + kbi, \ k \text{ um número real}$$

9. O conjunto de números reais $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, adição e multiplicação escalar definidas por

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{12} + b_{12} & a_{11} + b_{11} \\ a_{22} + b_{22} & a_{21} + b_{21} \end{pmatrix}$$
$$k\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{pmatrix}$$

10. O conjunto de todos os polinômios de grau 2

Nos Problemas 11-16, determine se o conjunto indicado é ou não é um subespaço do espaço vetorial $C(-\infty, \infty)$.

11. Todas as funções f de modo que $f(1) = 0$
12. Todas as funções f de modo que $f(0) = 1$
13. Todas as funções não-negativas f
14. Todas as funções f de modo que $f(-x) = f(x)$
15. Todas as funções diferenciáveis f
16. Todas as funções f da forma $f(x) = c_1 e^x + c_2 x e^x$

Nos Problemas 17-20, determine se o conjunto indicado é ou não é um subespaço do espaço vetorial indicado.

17. Polinômios da forma $p(x) = c_3 x^3 + c_1 x$; P_3
18. Polinômios p que são divisíveis por $x - 2$; P_2
19. Todos os vetores unitários; R^3
20. Funções f de modo que $\int_a^b f(x)\,dx = 0$; $C[a, b] = 0$
21. Em três dimensões, uma reta através da origem pode ser escrita como $S = \{(x, y, z) | x = at, y = bt, z = ct, a, b, c$ números reais$\}$. Com adição e multiplicação escalar igual para os vetores $\langle x, y, z \rangle$, mostre que S é um subespaço de R^3.

22. Em três dimensões, um plano através da origem pode ser escrito como $S = \{(x, y, z) | ax + by + cz = 0, a, b, c$ números reais$\}$. Mostre que S é um subespaço de R^3.

23. Os vetores $\mathbf{u}_1 = \langle 1, 0, 0\rangle$, $\mathbf{u}_2 = \langle 1, 1, 0\rangle$ e $\mathbf{u}_3 = \langle 1, 1, 1\rangle$ formam uma base para o espaço vetorial R^3.

    (a) Mostre que $\mathbf{u}_1$, $\mathbf{u}_2$ e $\mathbf{u}_3$ são linearmente independentes.

    (b) Escreva o vetor $\mathbf{a} = \langle 3, -4, 8\rangle$ como uma combinação linear de $\mathbf{u}_1$, $\mathbf{u}_2$ e $\mathbf{u}_3$.

24. Os vetores $p_1(x) = x + 1$, $p_2(x) = x - 1$ formam uma base para o espaço vetorial P_1.

    (a) Mostre que $p_1(x)$ e $p_2(x)$ são linearmente independentes.

    (b) Escreva o vetor $p(x) = 5x + 2$ como uma combinação linear de $p_1(x)$ e $p_2(x)$.

Nos Problemas 25-28, determine se os vetores indicados são linearmente independentes ou linearmente dependentes.

25. $\langle 4, -8\rangle, \langle -6, 12\rangle$ em R^2

26. $\langle 1, 1\rangle, \langle 0, 1\rangle, \langle 2, 5\rangle$ em R^2

27. $1, (x + 1), (x + 1)^2$ em P_2

28. $1, (x + 1), (x + 1)^2, x^2$ em P_2

Exercícios: Funções Ortogonais (Zill, Vol 3)

Nos Problemas 1-6, mostre que as funções apresentadas são ortogonais no intervalo indicado.

1. $f_1(x) = x,\ f_2(x) = x^2;\ [-2, 2]$
2. $f_1(x) = x^3,\ f_2(x) = x^2 + 1;\ [-1, 1]$
3. $f_1(x) = e^x,\ f_2(x) = xe^{-x} - e^{-x};\ [0, 2]$
4. $f_1(x) = \cos x,\ f_2(x) = \operatorname{sen}^2 x;\ [0, \pi]$
5. $f_1(x) = x,\ f_2(x) = \cos 2x;\ [-\pi/2, \pi/2]$
6. $f_1(x) = e^x,\ f_2(x) = \operatorname{sen} x;\ [\pi/4, 5\pi/4]$

Nos Problemas 7-12, mostre que o conjunto de funções apresentado é ortogonal no intervalo indicado. Determine a norma de cada função no conjunto.

7. $\{\operatorname{sen} x,\ \operatorname{sen} 3x,\ \operatorname{sen} 5x, \ldots\};\ [0, \pi/2]$
8. $\{\cos x,\ \cos 3x,\ \cos 5x, \ldots\};\ [0, \pi/2]$
9. $\{\operatorname{sen} nx\},\ n = 1, 2, 3, \ldots;\ [0, \pi]$
10. $\left\{\operatorname{sen}\frac{n\pi}{p}x\right\},\ n = 1, 2, 3, \ldots;\ [0, p]$
11. $\left\{1, \cos\frac{n\pi}{p}x\right\},\ n = 1, 2, 3, \ldots;\ [0, p]$
12. $\left\{1, \cos\frac{n\pi}{p}x, \operatorname{sen}\frac{m\pi}{p}x\right\},\ n = 1, 2, 3, \ldots,$
    $m = 1, 2, 3, \ldots;\ [-p, p]$

Exercícios: Série de Fourier (Zill, vol 3)

Nos Problemas 1-16, determine a série de Fourier de f no intervalo indicado.

1. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$

2. $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 2, & 0 \le x < \pi \end{cases}$

3. $f(x) = \begin{cases} 1, & -1 < x < 0 \\ x, & 0 \le x < 1 \end{cases}$

4. $f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 \le x < 1 \end{cases}$

5. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 \le x < \pi \end{cases}$

6. $f(x) = \begin{cases} \pi^2, & -\pi < x < 0 \\ \pi^2 - x^2, & 0 \le x < \pi \end{cases}$

7. $f(x) = x + \pi, \quad -\pi < x < \pi$

8. $f(x) = 3 - 2x, \quad -\pi < x < \pi$

9. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \operatorname{sen} x, & 0 \le x < \pi \end{cases}$

10. $f(x) = \begin{cases} 0, & -\pi/2 < x < 0 \\ \cos x, & 0 \le x < \pi/2 \end{cases}$

11. $f(x) = \begin{cases} 0, & -2 < x < -1 \\ -2, & -1 \le x < 0 \\ 1, & 0 \le x < 1 \\ 0, & 1 \le x < 2 \end{cases}$

12. $f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases}$

13. $f(x) = \begin{cases} 1, & -5 < x < 0 \\ 1 + x, & 0 \le x < 5 \end{cases}$

14. $f(x) = \begin{cases} 2 + x, & -2 < x < 0 \\ 2, & 0 \le x < 2 \end{cases}$

15. $f(x) = e^x, \quad -\pi < x < \pi$

16. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ e^x - 1, & 0 \le x < \pi \end{cases}$

17. Utilize o resultado do Problema 5 para mostrar

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

e

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots.$$

18. Use o Problema 17 para determinar uma série que dê o valor numérico de $\pi^2/8$.

19. Utilize o resultado do Problema 7 para mostrar

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

20. Utilize o resultado do Problema 9 para mostrar

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \frac{1}{7 \cdot 9} + \cdots$$

21. O **valor quadrático médio** (RMS)* de uma função $f(x)$ definida no intervalo (a, b) é definido como sendo

$$\text{RMS}(f) = \sqrt{\frac{\int_a^b f^2(x)\,dx}{b - a}}.$$

Considerando que a expansão em série de Fourier de f seja dada por (8), mostre que o valor RMS de f no intervalo $(-p, p)$ é definido como

$$\text{RMS}(f) = \sqrt{\tfrac{1}{4}a_0^2 + \tfrac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)},$$

onde a_0, a_n e b_n são os coeficientes de Fourier em (9), (10) e (11).

Exercícios: Séries Co-Seno e Seno (Zill, vol 3)

Nos Problemas 1-10, determine se a função é par, ímpar, ou nem par nem ímpar.

1. $f(x) = \operatorname{sen} 3x$
2. $f(x) = x \cos x$
3. $f(x) = x^2 + x$
4. $f(x) = x^3 - 4x$
5. $f(x) = e^{|x|}$
6. $f(x) = e^x - e^{-x}$
7. $f(x) = \begin{cases} x^2, & -1 < x < 0 \\ -x^2, & 0 \le x < 1 \end{cases}$
8. $f(x) = \begin{cases} x + 5, & -2 < x < 0 \\ -x + 5, & 0 \le x < 2 \end{cases}$
9. $f(x) = x^3, 0 \le x \le 2$
10. $f(x) = |x^5|$

Nos Problemas 11-24, expanda a função indicada em uma série apropriada de co-senos ou de senos.

11. $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$
12. $f(x) = \begin{cases} 1, & -2 < x < -1 \\ 0, & -1 < x < 1 \\ 1, & 1 < x < 2 \end{cases}$
13. $f(x) = |x|, -\pi < x < \pi$
14. $f(x) = x, -\pi < x < \pi$
15. $f(x) = x^2, -1 < x < 1$
16. $f(x) = x|x|, -1 < x < 1$
17. $f(x) = \pi^2 - x^2, -\pi < x < \pi$
18. $f(x) = x^3, -\pi < x < \pi$
19. $f(x) = \begin{cases} x - 1, & -\pi < x < 0 \\ x + 1, & 0 \le x < \pi \end{cases}$
20. $f(x) = \begin{cases} x + 1, & -1 < x < 0 \\ x - 1, & 0 \le x < 1 \end{cases}$
21. $f(x) = \begin{cases} 1, & -2 < x < -1 \\ -x, & -1 \le x < 0 \\ x, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases}$
22. $f(x) = \begin{cases} -\pi, & -2\pi < x < -\pi \\ x, & -\pi \le x < \pi \\ \pi, & \pi \le x < 2\pi \end{cases}$
23. $f(x) = |\operatorname{sen} x|, -\pi < x < \pi$
24. $f(x) = \cos x, -\pi/2 < x < \pi/2$

Exercícios: Séries Fourier – Forma Complexa (Zill, vol 3)

Nos Problemas 1-6, determine a série complexa de Fourier de f no intervalo indicado.

1. $f(x) = \begin{cases} -1, & -2 < x < 0 \\ 1, & 0 < x < 2 \end{cases}$

2. $f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 < x < 2 \end{cases}$

3. $f(x) = \begin{cases} 0, & -\frac{1}{2} < x < 0 \\ 1, & 0 < x < \frac{1}{4} \\ 0, & \frac{1}{4} < x < \frac{1}{2} \end{cases}$

4. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

5. $f(x) = x, 0 < x < 2\pi$

6. $f(x) = e^{-|x|}, -1 < x < 1$

7. Determine o espectro de frequência da onda periódica que seja a extensão periódica da função f no Problema 1.

8. Determine o espectro de frequência da onda periódica que seja a extensão periódica da função f no Problema 3.

Nos Problemas 9 e 10, esboce a onda periódica indicada. Determine o espectro de frequência de f.

9. $f(x) = 4 \operatorname{sen} x, 0 < x < \pi; f(x + \pi) = f(x)$
[Sugestão: Utilize (2).]

10. $f(x) = \begin{cases} \cos x, & 0 < x < \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases} \quad f(x + \pi) = f(x)$

11. (a) Mostre que $a_n = c_n + c_{-n}$ e $b_n = i(c_n + c_{-n})$.
(b) Utilize os resultados do item (a) e a série complexa de Fourier no Exemplo 1 para obter a expansão em série de Fourier de f.

12. A função f no Problema 1 é ímpar. Use a série complexa de Fourier para obter a expansão de f em série de Fourier de senos.